



EXACT OUTPUT TRACKING FOR THE ONE-DIMENSIONAL HEAT EQUATION AND APPLICATIONS TO THE INTERPOLATION PROBLEM IN GEVREY CLASSES OF ORDER

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ABSTRACT. This paper provides a complete characterization of the Dirichlet boundary outputs that can be exactly tracked in the one-dimensional heat equation with Neumann boundary control. The problem consists in describing the set of boundary traces generated by square-integrable controls over a finite or infinite time horizon. We show that these outputs form a precise functional space related to Gevrey regularity of order 2. In the infinite-time case, the trackable outputs are precisely those functions whose successive derivatives satisfy a weighted summability condition, which corresponds to specific Gevrey classes. For finite-time horizons, an additional compatibility condition involving the reachable space of the system provides a full characterization. The analysis relies on Fourier-Laplace transform, properties of Hardy spaces, the flatness method, and a new Plancherel-type theorem for Hilbert spaces of Gevrey functions. Beyond control theory, our results yield an optimal solution to the classical interpolation problem in Gevrey-2 classes, which improves results of Mitjagin on the optimal loss factor. The techniques developed here also extend to variants of the heat system with different boundary conditions or observation points.

1. Introduction.

1.1. Output tracking. Let us consider the one-dimensional heat problem with Neumann boundary conditions

$$\begin{cases} z_t(t, x) = z_{xx}(t, x), & 0 < x < 1, \quad t > 0, \\ z_x(t, 1) = u(t), \\ z_x(t, 0) = 0, \\ z(0, x) = 0, \\ y(t) = z(t, 0), \end{cases} \quad (1)$$

where $z = z(t, x)$ denotes the unknown state, $u = u(t)$ the control, and $y = y(t)$ is what we call the *output signal*. In the present article, we aim to obtain a sharp

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characterization of these outputs y when u ranges over $L^2(0, T)$. In other words, we look for a description of the set

$$\mathcal{Y}(0, T) := \{y \in L^1_{\text{loc}}[0, T] : u \in L^2(0, T)\},$$

where $T \in (0, \infty]$ is the horizon time. We refer to this as an *exact output tracking problem*.

In general, output tracking is a central problem in automatic control theory. From both theoretical and practical perspectives, the full state of a control system is often neither directly controllable nor observable, making full-state tracking unrealistic. In such situations, one seeks instead to track only a given *output* of the system, such as a boundary trace, as in the present example.

Furthermore, output tracking can be viewed as a preliminary step towards the more demanding problem of *output regulation*, which also involves disturbance rejection and robustness considerations. Most of the existing literature deals with asymptotic tracking achieved through (possibly dynamic) feedback laws, where the controlled output asymptotically approaches a reference trajectory as $t \rightarrow +\infty$ (see, for instance, [back] for related results on the one-dimensional heat equation).

In contrast, much less is known about *exact* or *approximate* output tracking achieved through open-loop controls, both for finite-dimensional linear systems and for controlled partial differential equations. For more on this finite-dimensional setting see [mesarovic], [sain'massey], [trentelman], [hautus], [tucsnak'rissel] and [zzz25]. In the context of the wave equation, exact output tracking in a closely related framework was investigated in [Z1, Z2]. Finally, for higher-dimensional generalizations of system (1), the weaker notion of *approximate output tracking*—where the desired tracking property is achieved up to an arbitrarily small error using an open-loop control—has been addressed in [MR4873205].

Concerning the exact tracking of (1), only a few results exist in the literature, which are all in the case of finite T . On the one hand, any $y \in \mathcal{Y}(0, T)$ satisfies $y \in C^\infty[0, T]$ together with the two conditions

$$y^{(k)}(0) = 0, \quad \forall k \in \mathbb{N}, \quad (2)$$

and

$$\exists C, R > 0, \quad \forall t \in [0, T], \quad \forall k \in \mathbb{N}, \quad |y^{(k)}(t)| \leq C \frac{(2k)!}{R^{2k}}. \quad (3)$$

Such a regularity result can be deduced from Gevrey's estimates [benedetto]. Moreover, in the above, the parameter R is uniform with respect to T and y . On the other hand, it has been established by Martin, Rosier and Rouchon in [reachable'martin] (see also Holmgren [holmgren] for connected results) using the flat-output method that a sufficient condition for a signal $\varphi \in C^\infty[0, T]$ to belong to $\mathcal{Y}(0, T)$ is that there is some $R > 1$ such that φ satisfies (2) together with (3).

Let us denote by $G^{2,R}[0, T]$ the set of those $\varphi \in C^\infty[0, T]$ satisfying the estimate (3), and $G^{2,R}_{(0)}[0, T]$ the set of those $\varphi \in G^{2,R}[0, T]$ which satisfy in addition (2). The above two results can be summed up as

$$\exists R \leq 1, \quad \forall \epsilon, T > 0, \quad G^{2,1+\epsilon}_{(0)}[0, T] \subset \mathcal{Y}(0, T) \subset G^{2,R}_{(0)}[0, T].$$

Our description of $\mathcal{Y}(0, T)$ will use Hilbert spaces of Gevrey functions similar to $G^{2,R}$. We denote by $C^\infty_{(0)}[0, \infty)$ the set of those $\varphi \in C^\infty[0, \infty)$ which satisfy in addition (2).

We start with the case $T = +\infty$, which is our first main result.

Theorem 1.1. *The set $\mathcal{Y}(0, \infty)$ consists of those $\varphi \in C_{(0)}^\infty[0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \left(\frac{\|\varphi^{(k+1)}\|_{L^2(0, \infty)}}{(2k)! 2^k (1+k)^{3/4}} \right)^2 < \infty. \quad (4)$$

The estimate (4) essentially means that φ is Gevrey of order 2 and radius $1/\sqrt{2}$. The constant $1/\sqrt{2}$ is no coincidence, and could have been guessed from the characterization of the reachable space of the heat equation and the flat-output method. We introduce these two theories below.

The reachable space for (1) is defined by

$$\mathcal{R}_T := \{z(T, \cdot) : u \in L^2(0, T)\}, \quad (T < \infty). \quad (5)$$

The set $\mathcal{R}_T$ has been studied in great detail in [fattorini·reachable, hartmann·orsoni, seidman, reachable·martin, sharp·reachable, reachable·heat, rkhs] where an explicit characterization was obtained. As shown in the cited references, the set $\mathcal{R}_T$ does not depend on T ; we denote it by $\mathcal{R}$. Moreover, it consists of those functions f that are holomorphic and even on the tilted square

$$\Omega := \{\zeta = a + ib \in \mathbb{C} : |a| + |b| < 1\},$$

and such that, moreover, $f' \in L^2(\Omega)$, where $f' = \partial f / \partial \zeta$ is the complex derivative. Note that such an f satisfies

$$\frac{\partial f}{\partial \bar{\zeta}} = 0, \quad \frac{\partial f}{\partial \zeta} \in L^2(\Omega),$$

hence, regarding f as a function of two real variables we have $\nabla f \in L^2(\Omega)$. Therefore, by standard procedures (see *e.g.* [mazya]), we deduce that f belongs to the Sobolev space $H^1(\Omega)$, where Ω is seen as a subset of $\mathbb{R}^2$. Thus,

$$\mathcal{R} = \{f \in \mathcal{H}(\Omega) \cap H^1(\Omega) : f \text{ even}\}, \quad (6)$$

where $\mathcal{H}(D)$ stands for the set of holomorphic functions on D , whenever D is an open subset of $\mathbb{C}$.

Now take a solution z of (1). It lies in $C^\infty((0, T) \times (0, 1))$, and can be extended as an even function of x , lying in $C^\infty((0, T) \times (-1, 1))$, which solves the heat equation on $(0, T) \times (-1, 1)$. The flat-output method is a general procedure to parameterize the state of a system as a power series of the successive time derivatives of a well-chosen output, we refer to [fliess] for a general overview. For the heat equation (1), this method has been implemented by developing z as power series of x , we refer to [holm·gren, laroche, reachable·martin]. In particular, for any $u \in L^2(0, T)$, the solution z of (1) writes

$$z(t, x) = \sum_{k=0}^{\infty} y^{(k)}(t) \frac{x^{2k}}{(2k)!}, \quad (7)$$

where for fixed $0 \leq t \leq T$, the power series converges for $|x|$ small enough.

From (7) and Cauchy's inequalities, we see that the largest R such that $y \in G^{2,R}[0, T]$ is also the largest R such that $z(t, x)$ is holomorphic with respect to x in $D(0, R)$. From (6), it follows that the largest such R is $R = 1/\sqrt{2}$, therefore, trackable functions are Gevrey of order 2 and radius $R \leq 1/\sqrt{2}$, and no better in general. Theorem 1.1 makes this more precise and more importantly provides a converse statement.

Let us now turn to the case $0 < T < \infty$. Clearly, any $y \in \mathcal{Y}(0, T)$ has an extension to $\mathcal{Y}(0, \infty)$ (extend the control u by zero for times $t > T$), which has to satisfy (4). Therefore, a necessary condition for $y \in \mathcal{Y}(0, T)$ is that

$$\sum_{k=0}^{\infty} \left(\frac{\|y^{(k+1)}\|_{L^2(0, T)}}{(2k)! 2^k (1+k)^{3/4}} \right)^2 < \infty. \quad (8)$$

Another necessary condition for y to belong to $\mathcal{Y}(0, T)$ is that $z(t, x)$ defined in (7) satisfies (1) (with the control law u generating the output y). From (5) and (7), this yields

$$\sum_{k=0}^{\infty} y^{(k)}(T) \frac{x^{2k}}{(2k)!} \in \mathcal{R}. \quad (9)$$

It turns out that the two conditions (8) and (9) are also sufficient. We denote by $C_{(0)}^{\infty}[0, T]$ the set of those $\varphi \in C^{\infty}[0, T]$ which satisfy in addition (2).

Theorem 1.2. *For any $0 < T < \infty$, the space $\mathcal{Y}(0, T)$ consists of those $y \in C_{(0)}^{\infty}[0, T]$ which satisfy the two conditions (8) and (9).*

Using cutoff functions that are Gevrey of order $1 < s < 2$, we easily deduce the following.

Corollary 1.3. *Let $0 < T < \infty$ and $\varphi \in C_{(0)}^{\infty}[0, T]$ satisfying (8) (where y is replaced by φ). Then for every $0 < \delta < T$, we have $\varphi \in \mathcal{Y}(0, T - \delta)$.*

The condition (8) does not imply (9); the following proposition provides a counterexample. Here and throughout, the symbol $\lesssim$ means the inequality holds up to a numerical multiplicative constant independent of the parameters; a similar meaning is used for $\gtrsim$.

Proposition 1.4. *There exists a sequence of real non-negative numbers $(a_k)_{k \in \mathbb{N}}$ such that:*

- *There exists $\varphi \in C^{\infty}[0, 1]$ such that*

$$\varphi^{(k)}(1^-) = a_k, \quad |\varphi^{(k)}(t)| \lesssim \frac{(2k)!}{\sqrt{1+k}}, \quad (k \geq 0, \quad 0 \leq t \leq 1). \quad (10)$$

- *There is no $\psi \in C^{\infty}[1, 2]$ satisfying (8) (with $(1, 2)$ in place of $(0, T)$) such that*

$$\psi^{(k)}(1^+) = a_k, \quad \forall k \geq 0. \quad (11)$$

Remark 1.5. We notice that Proposition 1.4 contradicts [reachable·martin], see Appendix A.

1.2. An intermediate result of interest. Let us sketch the proof of Theorem 1.1, in order to emphasize an intermediate result (Theorem 1.7 below) that is interesting in itself. For $u \in L^2(0, \infty)$, we apply the Laplace transform with respect to time in (1) to obtain

$$\hat{y}(s) = \frac{\hat{u}(s)}{\sqrt{s} \sinh \sqrt{s}}, \quad s \in \mathbb{C}_+,$$

where $\hat{\cdot}$ stands for the Laplace transform and $\mathbb{C}_+$ is the open right half-plane. This can be rewritten as

$$\hat{u}(s) = \phi(s) \hat{y}(s), \quad \phi(s) := \frac{\sinh \sqrt{s}}{\sqrt{s}}, \quad (12)$$

where $\hat{u}$ belongs to the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ and $1/\phi \in \mathcal{H}^\infty(\mathbb{C}_+)$, hence $\hat{y} \in \mathcal{H}^2(\mathbb{C}_+)$. Here, the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ (resp. $\mathcal{H}^\infty(\mathbb{C}_+)$) is defined as those $F \in \mathcal{H}(\mathbb{C}_+)$ such that

$$\sup_{0 < \sigma < \infty} \int_{\mathbb{R}} |F(\sigma + i\tau)|^2 d\tau < \infty, \quad \left(\text{resp. } \sup_{s \in \mathbb{C}_+} |F(s)| < \infty \right),$$

see [koosis] or [rudin]. From (12) we see that the description of $\mathcal{Y}(0, \infty)$ is equivalent, through the Laplace transform, to the description of these $F \in \mathcal{H}^2(\mathbb{C}_+)$ such that ϕF lies again in $\mathcal{H}^2(\mathbb{C}_+)$. Hardy functions can be characterized by their boundary values: a function $G \in \mathcal{H}(\mathbb{C}_+)$ with $L^2(\mathbb{R})$ boundary values $G(i\tau) = g(\tau)$, viz.

$$G(\sigma + i\tau) \xrightarrow[\sigma \rightarrow 0^+]{L^2(\mathbb{R}, d\tau)} g(\tau),$$

is such that $G \in \mathcal{H}^2(\mathbb{C}_+)$ if and only if the boundary value g satisfies

$$\text{spec } g := \text{supp } \mathcal{F}^{-1}g \subset [0, \infty). \quad (13)$$

In the above, the term $\text{spec } g$ is usually called the spectrum of g . In the present situation the function G has the specific form $G(s) = \phi(s)F(s)$, where $F \in \mathcal{H}^2(\mathbb{C}_+)$ and ϕ is entire with exponential order < 1 , i.e.

$$\exists 0 < \rho < 1, \quad \exists C > 0, \quad \forall s \in \mathbb{C}, \quad |\phi(s)| \leq C e^{C|s|^\rho}.$$

We shall use the following direct consequence of [foures].

Proposition 1.6. *Let ϕ be an entire function of order < 1 . Then, for every $F \in \mathcal{H}^2(\mathbb{C}_+)$, the product ϕF lies in $\mathcal{H}^2(\mathbb{C}_+)$ if and only if*

$$\int_{\mathbb{R}} |\phi(i\tau)F(i\tau)|^2 d\tau < \infty.$$

Indeed, set $G = \phi F$. If $\varphi(i\cdot)F(i\cdot) \in L^2(\mathbb{R})$, then G admits this function as its boundary value in $L^2_{\text{loc}}(\mathbb{R})$. Moreover, for some $0 < \rho < 1$,

$$|G(z)| \leq (1 + |F(z)|)C e^{C|z|^\rho},$$

so that [foures] applies with $K = F \in \mathcal{H}^2(\mathbb{C}_+)$. The converse follows from the usual boundary-value theorem for Hardy functions. Note in particular that the spectral condition (13) has disappeared. We readily obtain that

$$\int_{\mathbb{R}} |\mathcal{F}\hat{y}(\xi)\omega(\xi)|^2 d\xi < \infty, \quad \omega(\xi) = \frac{e^{\sqrt{|\xi|/2}}}{(1 + |\xi|)^{1/2}}, \quad (14)$$

characterizes $\mathcal{Y}(0, \infty)$. To translate (14) in the time domain, we will establish the following Plancherel-type theorem for certain Hilbert spaces of Gevrey functions. For parameters $s, R > 0$ and $\gamma \in \mathbb{R}$ define the Hilbert space

$$\hat{\mathcal{G}}_{s,R,\gamma} := L^2(\mathbb{R}, \omega(\xi)^2 d\xi), \quad \omega(\xi) = (1 + |\xi|)^\gamma e^{R|\xi|^{1/s}}.$$

Note that $\hat{\mathcal{G}}_{s,R,\gamma} \subset L^2(\mathbb{R}, d\xi)$ with continuous dense injection. The Fourier transforms of the elements of $\hat{\mathcal{G}}_{s,R,\gamma}$ turn out to be precisely those $\varphi \in C^\infty(\mathbb{R})$ such that

$$\|\varphi\|_{\hat{\mathcal{G}}_{s,R,\gamma}}^2 := \sum_{n=0}^{\infty} \left(\frac{\|\varphi^{(n)}\|_{L^2(\mathbb{R})}}{M_n} \right)^2 < \infty, \quad M_n := \frac{(ns)!}{R^{ns}} (1 + n)^{-s\gamma - 1/4}, \quad (15)$$

where $(ns)! := \Gamma(ns + 1)$. The above defines a Hilbert space $\mathcal{G}_{s,R,\gamma}$, which is continuously and densely embedded in $L^2(\mathbb{R}, dx)$.

Theorem 1.7. *For every $s, R > 0$ and $\gamma \in \mathbb{R}$, the Fourier transform is a topological isomorphism between $\mathcal{G}_{s,R,\gamma}$ and $\hat{\mathcal{G}}_{s,R,\gamma}$.*

Remark 1.8. As will be clear from the proof of Theorem 1.7, there exists a sequence (θ_n) of positive numbers satisfying

$$\exists 0 < \underline{\theta} \leq \bar{\theta} < \infty, \quad \forall n \in \mathbb{N}, \quad \underline{\theta} \leq \theta_n \leq \bar{\theta},$$

and such that if we replace M_n by $\theta_n M_n$ in the definition of $\mathcal{G}_{s,R,\gamma}$, then the Fourier transform is an isometry $\mathcal{G}_{s,R,\gamma} \rightarrow \hat{\mathcal{G}}_{s,R,\gamma}$. This fact will however not be used, and it will be more convenient for us to work with M_n given in (15).

To our knowledge, Theorem 1.7 is new even for $\gamma = 0$. Indeed, what is well-established is that a function $\varphi \in C_c^\infty(\mathbb{R})$ such that

$$\exists R, C > 0, \quad \forall k \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad |\varphi^{(k)}(x)| \leq C \frac{(ks)!}{R^{ks}},$$

will have a Fourier transform decaying like

$$\exists L, C' > 0, \quad \forall \xi \in \mathbb{R}, \quad |\mathcal{F}\varphi(\xi)| \leq C' e^{-L|\xi|^{1/s}},$$

and the converse also holds (see *e.g.* [gs2],[Rodino]). In previous works of which we are aware, a link between R and L can be made explicit, but it is never sharp.

1.3. Consequences for the interpolation problem in Gevrey-2 class. We now consider the interpolation problem in Gevrey classes¹ of order $s > 1$. For fixed $s > 1$, we consider a sequence (a_n) of complex numbers such that

$$\exists R_0, C > 0, \quad \forall n \in \mathbb{N}, \quad |a_n| \leq C \frac{(ns)!}{R_0^{ns}}. \quad (16)$$

We say that a function $\varphi \in C^\infty(\mathbb{R})$ interpolates (a_n) (at $t = 0$) if $\varphi^{(n)}(0) = a_n$ for every n . From Borel's Theorem² we know that such a function φ exists, regardless of the growth hypothesis made on (a_n) . In [carleson] Carleson studies the universal moment problem and establishes a sufficient condition for it to be solved. The moment problem is connected with the interpolation problem through the formula

$$\varphi^{(n)}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-i\xi)^n \mathcal{F}\varphi(-\xi) d\xi, \quad \mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\xi} \varphi(t) dt.$$

As a consequence of his result on the moment problem, Carleson obtains that one may always interpolate a sequence (a_n) satisfying (16) by a function φ which is Gevrey of order s , *i.e.* satisfying

$$\exists R_1, C' > 0, \quad \forall n \in \mathbb{N}, \quad \forall t \in \mathbb{R}, \quad |\varphi^{(n)}(t)| \leq C' \frac{(ns)!}{R_1^{ns}}. \quad (17)$$

For the applications in control theory we have in mind [reachable·martin, rosier, laurent], understanding how the parameter R_1 in (17) depends on the parameter R_0 in (16) is crucial in order to obtain sharp reachability results. Around the same time as Carleson's results were published, Mitjagin [mitjagin] independently showed that one can always take $R_1 = \rho_s R_0 - \epsilon$, where $\rho_s := \cos(\pi/2s) < 1$ and ϵ is any prescribed positive number. In [mitjagin], the interpolating function φ is

¹We refer to [bilodeau] for a historical survey. For more results on the interpolation problem, in particular on more general Denjoy-Carleman classes, see [wahde, petzsche, kiro].

²Borel published this result in 1895 within his thesis [Bobo], although it seems to have been proved first by Peano in 1884, see [borel].

defined on the smaller interval $[-1, 1]$, which is not essential, as can be seen by cut-off arguments, see **[reachable·martin]**. We shall refer to the numerical constant ρ_s as the loss factor of the interpolation problem, as it prescribes how one should relax the growth assumptions when passing from (16) to (17). The constant ρ_s was further shown by Mitjagin to be sharp in the sense that for every $\epsilon, R_0 > 0$ and $s > 1$, there exists a sequence (a_n) satisfying (16) which can be interpolated by no function φ satisfying (17) with $R_1 \geq \rho_s R_0 + \epsilon$. Mitjagin's proof is quite elegant, and the optimality of the constant ρ_s is deduced from its optimality in a Phragmén-Lindelöf principle. The paper **[mitjagin]** was originally written in Russian, and its translation is rather hard to access. Furthermore, in view of its size (4 pages), the published version seems to offer only a sketch of the proof rather than a fully detailed demonstration. Hence, for the sake of completeness, we give in Appendix A.2 a detailed proof of Mitjagin's solution of the interpolation problem.

As a consequence of our study of the trackable subspace $\mathcal{Y}(0, T)$ we can make more precise the relation between (a_n) and φ , when $s = 2$. The following result characterizes the range of the Borel operator $\mathcal{B}_{t_0}$, defined on smooth functions by $\mathcal{B}_{t_0}\varphi := (\varphi^{(n)}(t_0))_{n=0}^\infty$, over Hilbert spaces of Gevrey functions. We will use the following notation: for an interval $I \subset \mathbb{R}$, parameters $s, R > 0$ and $\gamma \in \mathbb{R}$, we let $\mathcal{G}_{s,R,\gamma}(I)$ be the Hilbert space defined similarly as in (15), with $L^2(I)$ in place of $L^2(\mathbb{R})$.

Proposition 1.9. *Let $I \subset \mathbb{R}$ be a nontrivial interval and t_0 lying in its interior. Let also $R > 0$ and $p \in \mathbb{Z}$. Then*

$$\mathcal{B}_{t_0}\mathcal{G}_{2,R,p}(I) = \left\{ (a_k) : \sum_{k=\max(0,-p)}^\infty a_{k+p} \frac{(\sqrt{2}R\zeta)^{2k}}{(2k)!} \in A^2(\Omega) \right\},$$

and

$$\mathcal{B}_{t_0}\mathcal{G}_{2,R,p-1/2}(I) = \left\{ (a_k) : \sum_{k=\max(0,-p)}^\infty a_{k+p} \frac{(\sqrt{2}R\zeta)^{2k+1}}{(2k+1)!} \in A^2(\Omega) \right\}.$$

In the above, $A^2(\Omega)$ stands for the Bergman space on Ω , that is the set of functions which are holomorphic and square integrable on Ω .

For an arbitrary sequence (a_n) , we deduce a formula for the largest $R > 0$ such that (a_n) can be interpolated by a function φ belonging to the set $G^{2,R}[-1, 1]$, which is defined similarly as in (3) with $[-1, 1]$ in place of $[0, T]$. To state the result we introduce the notation

$$R_a := \sup\{R > 0 : \exists \varphi \in G^{2,R}[-1, 1], \quad \forall n \in \mathbb{N}, \quad \varphi^{(n)}(0) = a_n\},$$

where it is understood that $R_a = 0$ if the above supremum runs over the empty set.

Corollary 1.10. *Let $a = (a_k)$ be an arbitrary sequence. Then*

$$R_a = \sup \left\{ R \geq 0 : \sum_{k=0}^\infty a_k \frac{(\sqrt{2}R\zeta)^{2k}}{(2k)!} \in A^2(\Omega) \right\}.$$

The rest of the paper is organized as follows. In Section 2 we show our tracking results Theorems 1.1 and 1.2, as well as the intermediate result Theorem 1.7 and finally Proposition 1.4. In Section 3 we explain how our methods can be applied to systems close to (1), having different boundary conditions or smoother control laws, or where the output is given by the Dirichlet trace at a given point of $(0, 1)$.

In Section 4 we show Proposition 1.9 and Corollary 1.10. In Appendix A we study the interpolation problem in Gevrey classes of order $s > 1$, by providing a detailed proof of Mitjagin's results [mitjagin].

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2. Exact tracking.

2.1. A Plancherel result in Gevrey classes. In this subsection, we show Theorem 1.7. Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\mathcal{F}f \in L^1_{\text{loc}}(\mathbb{R})$. Let $s, R > 0$ and $\gamma \in \mathbb{R}$. The proof is done by estimating the quantity

$$\int_{\mathbb{R}} |\mathcal{F}f(\xi)\omega(\xi)|^2 d\xi, \quad \omega(\xi) := (1 + |\xi|)^\gamma e^{R|\xi|^{1/s}},$$

whether it is finite or not. We will replace the weight ω by another weight ϖ of the form

$$\varpi(z) = \sum_{k=0}^{\infty} a_k z^{2k}, \quad a_k \geq 0,$$

which is such that $\omega \asymp \varpi$ on $\mathbb{R}$ and ϖ is an entire and even function on $\mathbb{C}$. Here and throughout, $\asymp$ means that we both have $\lesssim$ and $\gtrsim$. Let us assume provisionally that such a weight ϖ exists, we then compute

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F}f(\xi)\omega(\xi)|^2 d\xi &\asymp \int_{\mathbb{R}} |\mathcal{F}f(\xi)\varpi(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} \left| \mathcal{F}f(\xi) \sum_{k=0}^{\infty} a_k \xi^{2k} \right|^2 d\xi \\ &= \sum_{k,l=0}^{\infty} \int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 \xi^{2k} a_k \xi^{2l} a_l d\xi \\ &= \sum_{k,l=0}^{\infty} a_k a_l \int_{\mathbb{R}} |\mathcal{F}f^{(k+l)}(\xi)|^2 d\xi \\ &\asymp \sum_{k,l=0}^{\infty} a_k a_l \|f^{(k+l)}\|_{L^2(\mathbb{R})}^2 \\ &= \sum_{n=0}^{\infty} \|f^{(n)}\|_{L^2(\mathbb{R})}^2 \sum_{k=0}^n a_k a_{n-k}. \end{aligned} \tag{18}$$

We are thus left to find such a ϖ and to estimate the quantity

$$A_n := \sum_{k=0}^n a_k a_{n-k},$$

as $n \rightarrow \infty$. For $\alpha, \beta > 0$ we introduce the two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire on $\mathbb{C}$. It admits the following asymptotic expansion on the positive real axis:

$$E_{\alpha,\beta}(x) \sim \frac{1}{\alpha} x^{\frac{1-\beta}{\alpha}} \exp x^{1/\alpha}, \quad x \rightarrow +\infty,$$

see [mittag'leffler].

A particular case. Let us assume that $\gamma < 0$, $R = 1$, and put

$$\varpi(z) := E_{\alpha,\beta+1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(\alpha k + \beta + 1)}, \quad \frac{1}{s} = \frac{2}{\alpha}, \quad \gamma = -2\frac{\beta}{\alpha}, \quad (19)$$

so that ϖ is entire and even on $\mathbb{C}$, bounded below by a positive constant on $\mathbb{R}$, and admits the asymptotic expansion

$$\varpi(\xi) \sim \frac{1}{\alpha} |\xi|^\gamma \exp |\xi|^{1/s}, \quad \xi \rightarrow \pm\infty.$$

To estimate A_n we start by writing Stirling's formula as

$$\frac{1}{a_k} = \Gamma(\alpha k + \beta + 1) \asymp (1+k)^{\beta+1/2} \left(\frac{\alpha k}{e}\right)^{\alpha k}, \quad \forall k \in \mathbb{N}.$$

Note that we write $(1+k)^{\beta+1/2}$ rather than $k^{\beta+1/2}$, so that the right-hand side of the above estimate never vanishes. We obtain, for $0 \leq k \leq n$ and $n \geq 1$,

$$\begin{aligned} \frac{1}{a_k a_{n-k}} &\asymp [(1+k)(1+n-k)]^{\beta+1/2} \left(\frac{\alpha k}{e}\right)^{\alpha k} \left(\frac{\alpha(n-k)}{e}\right)^{\alpha(n-k)} \\ &= n^{1+2\beta} \left[\left(\frac{1}{n} + \frac{k}{n}\right) \left(\frac{1}{n} + 1 - \frac{k}{n}\right) \right]^{\beta+1/2} \left(\frac{n\alpha}{e}\right)^{\alpha n} \times \\ &\quad \times \left[\left(\frac{k}{n}\right)^{\alpha k/n} \left(1 - \frac{k}{n}\right)^{\alpha(1-\frac{k}{n})} \right]^n \\ &= \left(\frac{n\alpha}{e}\right)^{\alpha n} n^{1+2\beta} g_n(k/n)^{-1} h(k/n)^n, \end{aligned}$$

with

$$g_n(x) = \left[\left(\frac{1}{n} + x\right) \left(\frac{1}{n} + 1 - x\right) \right]^{-\beta-1/2}, \quad h(x) = x^{\alpha x} (1-x)^{\alpha(1-x)}.$$

This brings

$$A_n \asymp \left(\frac{e}{\alpha n}\right)^{\alpha n} \frac{1}{n^{2\beta}} \cdot \frac{1}{n} \sum_{k=0}^n g_n(k/n) h(k/n)^{-n}, \quad (n \geq 1), \quad (20)$$

which reduces the estimation of A_n to that of the second factor on the right-hand side of (20). We write

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^n g_n(k/n) h(k/n)^{-n} &= g_n(1/2) \frac{1}{n} \sum_{k=0}^n h(k/n)^{-n} + \\ &\quad + \frac{1}{n} \sum_{k=0}^n (g_n(k/n) - g_n(1/2)) h(k/n)^{-n}. \end{aligned} \quad (21)$$

To estimate the above right-hand side, we begin by estimating the first of the two sums. We write

$$\frac{1}{n} \sum_{k=0}^n h(k/n)^{-n} = \frac{1}{n} \sum_{k=0}^n e^{-n \log h(k/n)},$$

which looks like a discrete analog of the integral

$$\int_0^1 e^{-n \log h(x)} dx.$$

For the above integral, the asymptotic behaviour is well known by Laplace's method. Following **[laplace discrete]**, we rely on the following Lemma.

Lemma 2.1. *Let $u : [0, 1] \rightarrow \mathbb{R}$ be in $C^2(0, 1) \cap C[0, 1]$ with global strict minimum at $x_0 \in (0, 1)$, for which $u''(x_0) > 0$. Then*

$$\frac{1}{n} \sum_{k=0}^n e^{-nu(k/n)} \sim \sqrt{\frac{2\pi}{u''(x_0)n}} e^{-nu(x_0)}, \quad n \rightarrow \infty.$$

The proof of this Lemma is elementary but rather lengthy, we defer it to the Appendix B. Applying this result with $u = \log h$ we obtain the following asymptotic behaviour, for which we recall that $h(1/2) = 2^{-\alpha}$,

$$\frac{1}{n} \sum_{k=0}^n h(k/n)^{-n} \asymp \frac{2^{\alpha n}}{\sqrt{n}}, \quad (n \geq 1).$$

Because $g_n(1/2) \asymp 1$, we deduce

$$g_n(1/2) \frac{1}{n} \sum_{k=0}^n h(k/n)^{-n} \asymp \frac{2^{\alpha n}}{\sqrt{n}}, \quad (n \geq 1). \quad (22)$$

Coming back to (21), we see that (22) may be plugged into (20) as soon as

$$\frac{1}{n} \sum_{k=0}^n (g_n(k/n) - g_n(1/2)) h(k/n)^{-n} = o\left(\frac{2^{\alpha n}}{\sqrt{n}}\right), \quad n \rightarrow \infty. \quad (23)$$

To show that (23) holds, we introduce a sequence $(\epsilon_n)_{n \in \mathbb{N}^*}$ with $0 < \epsilon_n < 1/2$ for any $n \in \mathbb{N}^*$, to be specified later on, and compute

$$\begin{aligned} & \left| \frac{1}{n} \sum_{k=0}^n (g_n(k/n) - g_n(1/2)) h(k/n)^{-n} \right| \\ & \leq \frac{1}{n} \sum_{\left| \frac{k}{n} - \frac{1}{2} \right| \leq \epsilon_n} |g_n(k/n) - g_n(1/2)| h(k/n)^{-n} \\ & \quad + \frac{1}{n} \sum_{\left| \frac{k}{n} - \frac{1}{2} \right| > \epsilon_n} |g_n(k/n) - g_n(1/2)| h(k/n)^{-n}. \end{aligned} \quad (24)$$

We estimate the two terms of the right-hand side of (24) separately. For the second one, we observe that g_n is positive on $[0, 1]$, where it reaches its maximal value at $x = 0$ and $x = 1$, for which

$$g_n(0) \sim n^{\beta+1/2}, \quad n \rightarrow \infty.$$

Moreover,

$$\min_{|x-1/2| \geq \epsilon_n} h(x) = h(1/2 + \epsilon_n),$$

hence

$$\frac{1}{n} \sum_{\left| \frac{k}{n} - \frac{1}{2} \right| > \epsilon_n} |g_n(k/n) - g_n(1/2)| h(k/n)^{-n} \lesssim n^{\beta+1/2} h(1/2 + \epsilon_n)^{-n}, \quad n \geq 1.$$

Recalling $h(1/2) = 2^{-\alpha}$, we deduce that

$$n^{\beta+1/2}h(1/2 + \epsilon_n)^{-n} = o\left(\frac{2^{\alpha n}}{\sqrt{n}}\right), \quad n \rightarrow \infty,$$

holds if and only if

$$n[\log h(1/2 + \epsilon_n) - \log h(1/2)] - (\beta + 1) \log n \rightarrow \infty, \quad n \rightarrow \infty$$

and by log convexity of h , there exists some constant $\kappa > 0$ such that

$$\log h(1/2 + \epsilon_n) - \log h(1/2) \geq \kappa \epsilon_n^2.$$

We take

$$\epsilon_n = \frac{1}{n^\mu}, \quad 0 < \mu < 1/2,$$

so that the second term on the right-hand side of (24) has the appropriate size. For the first term, we write

$$g_n(x) = \frac{1}{i_n(x)^{\beta+1/2}}, \quad i_n(x) = \left(\frac{1}{n} + x\right) \left(\frac{1}{n} + 1 - x\right),$$

and compute for $|x - 1/2| \leq \delta$

$$|i_n(x) - 1/4| \leq |i_n(x) - i_n(1/2)| + |i_n(1/2) - 1/4| \leq \delta^2 + \frac{2}{n}.$$

Therefore, for small δ and large n , the term $i_n(x)$ is close to $1/4$, around which the function $x \mapsto x^{-\beta-1/2}$ is Lipschitz. We thus put $\delta = \epsilon_n$, so that for n large enough

$$\frac{1}{n} \sum_{\left|\frac{k}{n} - \frac{1}{2}\right| \leq \epsilon_n} |g_n(k/n) - g_n(1/2)| h(k/n)^{-n} \lesssim \left(\epsilon_n^2 + \frac{1}{n}\right) 2^{\alpha n}.$$

We take $1/4 < \mu < 1/2$ in the definition of ϵ_n so that the above contribution is negligible compared with $2^{\alpha n}/\sqrt{n}$ as $n \rightarrow \infty$. This shows that (23) holds.

We may thus plug (22) in (20), which yields

$$A_n \asymp \left(\frac{2e}{\alpha n}\right)^{\alpha n} \frac{1}{n^{2\beta+1/2}}, \quad (n \geq 1).$$

From (18) and (15), we see that the sequence (M_n) should satisfy $M_n \asymp 1/\sqrt{A_n}$, and using (19) we compute

$$\frac{1}{\sqrt{A_n}} \asymp \left(\frac{ns}{e}\right)^{ns} n^{-s\gamma+1/4}, \quad (n \geq 1).$$

This ends the proof of Theorem 1.7, under the additional assumptions $R = 1$ and $\gamma < 0$.

General case. We extend the result to the cases of $R > 0$ and $\gamma \in \mathbb{R}$ arbitrary by acting on f . The case of general $R > 0$ is easily deduced from the case $R = 1$ by replacing $f(x)$ by $f(R^s x)$, so, without loss of generality, we may assume that $R = 1$. To allow γ to be non-negative, we pick $p \in \mathbb{N}$ large enough so that $\tilde{\gamma} := \gamma - p < 0$. We then observe that for every $f \in \mathcal{S}'(\mathbb{R})$ with $\mathcal{F}f \in L^1_{\text{loc}}(\mathbb{R})$, there holds

$$\int_{\mathbb{R}} |\mathcal{F}f(\xi)| e^{|\xi|^{1/s}} (1 + |\xi|)^\gamma d\xi \asymp \sum_{k=0}^p \int_{\mathbb{R}} |\mathcal{F}f^{(k)}(\xi)| e^{|\xi|^{1/s}} (1 + |\xi|)^{\tilde{\gamma}} d\xi$$

$$\begin{aligned}
& \asymp \sum_{k=0}^p \|f^{(k)}\|_{\mathcal{G}_{s,1,\tilde{\gamma}}}^2 \\
& = \sum_{k=0}^p \sum_{n=0}^{\infty} \left(\frac{\|f^{(k+n)}\|_{L^2(\mathbb{R})}}{M_n} \right)^2 \\
& \asymp \sum_{n=0}^{\infty} \left(\frac{\|f^{(n)}\|_{L^2(\mathbb{R})}}{N_n} \right)^2,
\end{aligned}$$

where

$$M_n = \left(\frac{ns}{e} \right)^{ns} (1+n)^{-s\tilde{\gamma}+1/4},$$

and (N_n) is any positive sequence with

$$N_n \asymp M_{n-p}, \quad n \geq p.$$

For instance, we can take

$$N_n := \left(\frac{ns}{e} \right)^{ns} (1+n)^{-s\gamma+1/4}.$$

This shows that

$$\int_{\mathbb{R}} |\mathcal{F}f(\xi)| e^{|\xi|^{1/s}} (1+|\xi|)^{\gamma} d\xi \asymp \|f\|_{\mathcal{G}_{s,1,\gamma}}^2,$$

which ends the proof of Theorem 1.7.

2.2. Proof of the main results. For system (1) we consider transposition solutions [Coron].

Definition 2.2. Let $u \in L^2(0, \infty)$, a solution of (1) is $z \in C([0, \infty); L^2(0, 1))$ such that for all $0 < T < \infty$ and all $\varphi^T \in L^2(0, 1)$, denoting $\varphi(t, x)$ the solution of

$$\begin{cases} -\varphi_t(t, x) &= \varphi_{xx}(t, x), & 0 < t < T, & 0 < x < 1, \\ \varphi_x(t, 1) &= 0, \\ \varphi_x(t, 0) &= 0, \\ \varphi(T, x) &= \varphi^T(x), \end{cases}$$

we have

$$\int_0^1 z(T, x) \varphi^T(x) dx = \int_0^T \varphi(t, 1) u(t) dt.$$

It is easy to see that (1) is then well-posed in the sense that for all $u \in L^2(0, \infty)$, there exists a unique solution z to (1), and that

$$\forall 0 < T < \infty, \quad \exists C > 0, \quad \forall u \in L^2(0, \infty), \quad \|z\|_{C([0, T]; L^2(0, 1))} \leq C \|u\|_{L^2(0, T)}.$$

Moreover, introducing the spectral decomposition of the Neumann Laplacian

$$\lambda_j = (j\pi)^2, \quad e_j(x) = \begin{cases} 1 & j = 0, \\ \sqrt{2} \cos(j\pi x) & j \geq 1, \end{cases}, \quad (j \in \mathbb{N})$$

one easily verifies that z assumes the representation

$$z(t) = \int_0^t \sum_{j=0}^{\infty} e^{-\lambda_j(t-\sigma)} e_j(1) u(\sigma) e_j d\sigma, \quad (25)$$

where, for fixed t , there holds

$$\int_0^t \left\| \sum_{j=0}^{\infty} e^{-\lambda_j(t-\sigma)} e_j(1) u(\sigma) e_j \right\|_{L^2(0,1)} d\sigma < \infty.$$

Thus, (25) is well-defined as a Bochner integral in $L^2(0, 1)$. Extending z evenly with respect to x and by zero for negative times, one obtains a distributional solution of the heat equation on $\mathbb{R} \times (-1, 1)$. By hypoellipticity of the heat operator $\partial_t - \partial_{xx}$, it follows that

$$z \in C^\infty([0, \infty) \times (-1, 1)).$$

Hence, the output $y(t) = z(t, 0)$ makes sense as a pointwise evaluation and belongs to $C_{(0)}^\infty[0, \infty)$.

Proof of Theorem 1.1. In what follows all the signals defined on $(0, \infty)$ are extended by 0 on $(-\infty, 0)$. Assume first that $y \in \mathcal{Y}(0, \infty)$, let us show that y is Laplace transformable and its Laplace transform satisfies (12). A formal proof is done by taking the Laplace transform in time in (1) (without checking whether it is legitimate or not), solving the resulting second order differential equation in space, using the boundary conditions given by the second and third lines of (1) and finally evaluating at $x = 0$.

The above method will be applied in §3.1.1 to another system. For (1) it will be more convenient to work with the representation formula (25). The representation formula (25) may be evaluated at $x = 0$, yielding

$$y(t) = z(t, 0) = \int_0^t k(t - \sigma) u(\sigma) d\sigma, \quad k(t) := 1 + 2 \sum_{j=1}^{\infty} e^{-(j\pi)^2 t} (-1)^j,$$

where the defined kernel k is extended by 0 for $t < 0$. The function k may be rewritten using the Poisson summation formula as

$$k(t) = \frac{1}{\sqrt{\pi t}} \sum_{m=-\infty}^{+\infty} e^{-\frac{(m+1/2)^2}{t}}.$$

Combining the two representations of k , we arrive at $k \in C_{(0)}^\infty[0, \infty) \cap L^\infty(0, \infty)$. The last fact can be used as an alternative proof of $y \in C_{(0)}^\infty[0, \infty)$. The signals k, u, y are therefore all Laplace transformable, with abscissa of absolute convergence no greater than 0. The convolution rule further applies:

$$\hat{y}(s) = \hat{k}(s) \hat{u}(s), \quad s \in \mathbb{C}_+,$$

where an explicit computation gives that

$$\hat{k}(s) = \sum_{m=-\infty}^{\infty} \mathcal{L} \left(\frac{1}{\sqrt{\pi t}} e^{-\frac{(m+1/2)^2}{t}} \right) (s) = \frac{1}{\sqrt{s}} \sum_{m=-\infty}^{\infty} e^{-2|m+1/2|\sqrt{s}} = \frac{1}{\sqrt{s}} \frac{2e^{\sqrt{s}}}{e^{2\sqrt{s}} - 1}.$$

In the above computation, the interchange of the sum and the integral is justified by considering first $s > 0$, which yields only positive quantities in the summand. We deduce that the transfer function of (1) is given by

$$\hat{k}(s) = \frac{1}{\sqrt{s} \sinh \sqrt{s}},$$

whence

$$\hat{y}(s) = \frac{\hat{u}(s)}{\sqrt{s} \sinh \sqrt{s}},$$

which rewrites as

$$\hat{u}(s) = \sqrt{s} \sinh \sqrt{s} \hat{y}(s) = \frac{\sinh \sqrt{s}}{\sqrt{s}} \hat{y}(s) = \phi(s) \hat{y}(s), \quad (26)$$

where we have denoted

$$\phi(s) := \frac{\sinh \sqrt{s}}{\sqrt{s}}. \quad (27)$$

Note that $1/\phi$ is bounded on $\mathbb{C}_+$, so that $\hat{y} \in \mathcal{H}^2(\mathbb{C}_+)$ and $\dot{y} \in L^2(0, \infty)$. In particular, the Laplace transform $\hat{y}$ has the L^2 boundary value $\hat{y}(i\xi) = \mathcal{F}\dot{y}(\xi)$. Now, coming back to

$$\phi \hat{y} = \hat{u} \in \mathcal{H}^2(\mathbb{C}_+),$$

we take advantage of the above function having square-integrable boundary values to deduce that $\dot{y}$ satisfies (14). From Proposition 1.6, the previous reasoning can be reversed, so that

$$\mathcal{Y}(0, \infty) = \{y \in \mathcal{S}'(\mathbb{R}) : \text{supp } y \subset [0, \infty), \quad \dot{y} \text{ satisfies (14)}\}.$$

The rest of the proof of Theorem 1.1 then follows from the Gevrey-Plancherel Theorem 1.7. $\square$

Proof of Theorem 1.2. It is enough to show that any function $\varphi \in C_{(0)}^\infty[0, T]$ satisfying the two conditions (8) and (9) lies in $\mathcal{Y}(0, T)$. Indeed, let φ be such a function, from (9) there is a control $u \in L^2(0, T)$ such that

$$z(T, x) = \sum_{k=0}^{\infty} \varphi^{(k)}(T^-) \frac{x^{2k}}{(2k)!}, \quad 0 < x < 1,$$

where z is the solution of (1) with control u . In particular, we have $\partial_t^k z(T, 0) = \varphi^{(k)}(T^-)$ for every $k \in \mathbb{N}$. We then extend the control u by 0 on (T, ∞) , which extends z for times $t > T$ as a $C^\infty([0, \infty) \times (-1, 1))$. Consider then

$$\psi(t) := \begin{cases} \varphi(t), & 0 < t < T, \\ z(t, 0), & T < t < \infty. \end{cases}$$

By Theorem 1.1, the function ψ lies in $\mathcal{Y}(0, \infty)$. Therefore, its restriction to $(0, T)$ lies in $\mathcal{Y}(0, T)$. This ends the proof of Theorem 1.2. $\square$

Remark 2.3. For an initial datum $z^0 \in L^2(0, 1)$, the solution z of

$$\begin{cases} z_t(t, x) &= z_{xx}(t, x), \quad 0 < x < 1, \quad t > 0, \\ z_x(t, 1) &= 0, \\ z_x(t, 0) &= 0, \\ z(0, x) &= z^0(x), \end{cases} \quad (28)$$

is of class $C^\infty((0, \infty) \times (-1, 1))$ and the output $y(t) = z(t, 0)$ is in $L^p(0, T)$ for all $1 \leq p < 4$ and $0 < T < \infty$. The value of p cannot be increased to $p > 4$. This however has no impact on the proof of Theorem 1.2, as therein we only consider such outputs for initial data in the reachable space, which are seen to be exactly shifted outputs of the system (1) (with a control and zero initial data). In other words, we do not need the existence of the output y for (28).

Proof of Corollary 1.3. Let $\varphi \in C_{(0)}^\infty[0, T]$ satisfying (8) and $0 < \delta < T$. Introduce $\chi \in C^\infty[0, T]$ such that

$$|\chi^{(n)}(t)| \leq C^{n+1} n^{ns}, \quad \chi(t) = 1 \quad (0 \leq t \leq T - \delta), \quad \chi(t) = 0 \quad (T - \delta/2 \leq t \leq T),$$

where $C > 0$ and $1 < s < 2$ are independent of $n \in \mathbb{N}$ and $t \in [0, T]$. Reasoning similarly to **[reachable·martin]** we see that $\chi\varphi$ satisfies again (8). Moreover, since $\chi\varphi$ is flat at $t = T$ it also satisfies (9). Thus from Theorem 1.2 we have $\chi\varphi \in \mathcal{Y}(0, T)$. Therefore, the restriction of $\chi\varphi$ to $[0, T - \delta]$ belongs to $\mathcal{Y}(0, T - \delta)$, which ends the proof. $\square$

2.3. Proof of Proposition 1.4. In this subsection we prove Proposition 1.4. We will rely on the following Lemma taken from **[reachable·martin]**, which is proved using the Borel summation **[ramis]**.

Lemma 2.4. *Let (a_n) be a sequence such that the power series*

$$f(z) := \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!(n+1)!} z^n,$$

has a nonzero radius of convergence. Assume that f extends to a holomorphic function on a neighborhood of $\mathbb{R}_-$ in $\mathbb{C}$ and that

$$\exists C > 0, \quad \forall n \geq 0, \quad \forall x < 0, \quad |f^{(n)}(x)| \leq C|f^{(n)}(0)|. \quad (29)$$

Then there exists $\varphi \in C^\infty[-1, 0]$ such that

$$\varphi^{(n)}(0) = a_n, \quad |\varphi^{(n)}(t)| \lesssim |a_n|, \quad (n \geq 0, \quad -1 \leq t \leq 0).$$

Proof of Proposition 1.4. Following **[reachable·martin]**, we consider the function

$$f(z) := \frac{1}{1-z},$$

which satisfies the hypotheses of Lemma 2.4. Hence there exists $\phi \in C^\infty[-1, 0]$ such that

$$\phi^{(n)}(0) = b_n, \quad |\phi^{(n)}(t)| \lesssim b_n, \quad (n \geq 0, \quad -1 \leq t \leq 0),$$

where $b_{n+1} := n!(n+1)!$ and $b_0 = 1$. Moreover, without loss of generality, we can assume that ϕ is flat at -1 , by multiplying ϕ by an appropriate Gevrey cut-off of order less than 2, with value 1 in a neighbourhood of 0, and flat at -1 .

By Stirling's formula,

$$b_n \asymp \frac{(2n)!}{\sqrt{1+n4^n}}.$$

We then consider $a_n := 4^n b_n$. By scaling, there exists $\varphi \in C^\infty[-1, 0]$ such that

$$\varphi^{(n)}(0) = a_n, \quad |\varphi^{(n)}(t)| \lesssim \frac{(2n)!}{\sqrt{1+n}}, \quad (n \geq 0, \quad -1 \leq t \leq 0).$$

Assume by contradiction that there exists some $\psi \in C^\infty[0, 1]$ such that for all $n \in \mathbb{N}$, $\psi^{(n)}(0) = a_n$, and (8) holds with $T = 1$. The sequence (a_n) is then the sequence of successive derivatives at an interior point of a $\mathcal{G}_{2,1/\sqrt{2},-1/2}$ function. Hence, by Corollary 1.3 we deduce

$$\sum_{n=0}^{\infty} a_n \frac{\zeta^{2n}}{(2n)!} \in \mathcal{R}.$$

However, a sharper estimate of b_n yields

$$\frac{a_n}{(2n)!} = \sqrt{\frac{\pi}{n}} + O\left(\frac{1}{n^{3/2}}\right),$$

where, for any bounded sequence (c_n) , the series $\sum c_n \zeta^{2n}/n^{3/2}$ converges absolutely in $A^2(D(0, 1))$. We thus deduce that the function

$$\text{Li}_{-1/2}(\zeta^2) = \sum_{n=1}^{\infty} \frac{\zeta^{2n}}{n^{-1/2}},$$

belongs to $A^2(\Omega)$. For every $s \in \mathbb{R}$, the function

$$\text{Li}_s(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n^s},$$

is a special function often called the polylogarithm, or Jonquière's function, and is a special case of Lerch's function. It is defined by a power series with radius of convergence 1 and has the following asymptotic behaviour

$$\text{Li}_s(\zeta^2) = \frac{\Gamma(1-s)2^{s-1}}{(1-\zeta)^{1-s}} + O(1), \quad \zeta \rightarrow 1, \quad \zeta \in D(1/2, 1/2), \quad (30)$$

for every $s \neq 1, 2, 3, \dots$, see [transcendental]. It is then clear that $\text{Li}_{-1/2}(\zeta^2)$ cannot lie in $A^2(\Omega)$, which is the sought contradiction. $\square$

We end this Section with two consequences of Proposition 1.4. First, we formulate below the extension problem for Gevrey functions of order 2, to be compared with Definition A.1.

Definition 2.5. We say that the extension problem (for $s = 2$) is solvable with loss $\rho \in (0, 1)$ if: for every function $\phi \in C^\infty[-1, 1]$ such that

$$\exists C, R_0 > 0, \quad \forall n \in \mathbb{N}, \quad \forall t \in [-1, 1], \quad |\phi^{(n)}(t)| \leq C \frac{(2n)!}{R_0^{2n}},$$

and for every $\epsilon > 0$ with $\epsilon < \rho R_0$, there exists a function $\varphi \in C^\infty[-2, 2]$ such that

$$\forall t \in [-1, 1], \quad \phi(t) = \varphi(t),$$

and

$$\exists C' > 0, \quad \forall n \in \mathbb{N}, \quad \forall t \in [-2, 2], \quad |\varphi^{(n)}(t)| \leq C' \frac{(2n)!}{(\rho R_0 - \epsilon)^{2n}}.$$

It is clear that if the extension problem can be solved with some loss factor ρ , then it can also be solved with any $\tilde{\rho} \leq \rho$. This allows us to define ρ^{Ext} as the optimal (*i.e.* maximal) loss factor ρ such that the extension problem is solvable with loss ρ . Recall that from Mitjagin's theorem the interpolation problem can be solved for Gevrey functions of order 2, the optimal loss factor being $\rho^{Carl} = 1/\sqrt{2}$. Proposition 1.4 shows that $\rho^{Ext} \leq 1/\sqrt{2}$ and it is easy to see that $\rho^{Ext} \geq \rho^{Carl}$, hence $\rho^{Ext} = \rho^{Carl}$. In other words: there is no way to extend functions in the Gevrey class of order 2 which yields a better loss factor than using the solution of the interpolation problem.

Secondly, we point out that Proposition 1.9 describes the range of the Borel operator, over some Hilbert spaces of Gevrey-2 functions on some interval I , when the point t_0 at which the derivatives are taken is an interior point of I . Such a description is harder when t_0 is a boundary point of I , in fact we have the following.

Proposition 2.6. Let $R > 0$ and denote $\mathcal{B}_{x_0}^R$ the Borel operator acting on the set $G^{2,R}[-1, 1]$, at the point $x_0 \in [-1, 1]$. We then have:

- For every $x_0 \in (-1, 1)$, $\text{Range } \mathcal{B}_{x_0}^R = \text{Range } \mathcal{B}_0^R \subset \text{Range } \mathcal{B}_{-1}^R \cap \text{Range } \mathcal{B}_1^R$.

- For every $\epsilon > 0$ with $\epsilon < R/\sqrt{2}$, there holds $\text{Range } \mathcal{B}_{\pm 1}^R \subset \text{Range } \mathcal{B}_{\mp 1}^{R/\sqrt{2}-\epsilon}$.
- For every $\epsilon > 0$, we do not have $\text{Range } \mathcal{B}_{\pm 1}^R \subset \text{Range } \mathcal{B}_{\mp 1}^{R/\sqrt{2}+\epsilon}$.

Proof. The first item is trivial, the second one is by Mitjagin's Theorem A.3, and the third item is a consequence of Proposition 1.4. $\square$

3. Similar systems.

3.1. Other boundary conditions.

3.1.1. *Neumann-to-Neumann.* Let us consider the system

$$\begin{cases} z_t(t, x) = z_{xx}(t, x), & 0 < x < 1, \quad t > 0, \\ z_x(t, 1) = u(t), \\ z(t, 0) = 0, \\ z(0, x) = z^0(x), \\ y(t) = z_x(t, 0), \end{cases} \quad (31)$$

which we call “Neumann-to-Neumann” because the control is through the Neumann action and the output is the Neumann trace. Let us first disregard the output y ; as for (1) we consider transposition solutions and it is easy to check that (31) is well-posed on the state space $L^2(0, 1)$. When $z^0 = 0$ and $u \in L^2(0, \infty)$, extending the solution z of (31) as an odd function of $x \in (-1, 1)$ and using the hypoellipticity of the heat equation one sees that $z \in C^\infty([0, \infty) \times (-1, 1))$. Hence, when $z^0 = 0$, the output $y(t) = z_x(t, 0)$ is just a pointwise evaluation. When $u = 0$ and $z^0 \in L^2(0, 1)$, one sees that $z \in C^\infty((0, \infty) \times (-1, 1))$ and $z_x(\cdot, 0) \in L^1(0, \infty)$, but we will not use this fact.

Now, let $u \in L^2(0, \infty)$, we compute the Laplace transform of y . To this aim, we first take $u \in H^1(0, \infty)$ such that $u(0) = 0$. We consider the function

$$\xi(t, x) = z(t, x) - x(x-1)u(t),$$

which is continuous in both variables and solves

$$\begin{cases} \xi_t(t, x) = \xi_{xx}(t, x) + f(t, x), & 0 < x < 1, \quad t > 0, \\ \xi_x(t, 1) = 0, \\ \xi(t, 0) = 0, \\ \xi(0, x) = 0, \end{cases},$$

where

$$f(t, x) := -x(x-1)\dot{u}(t) + 2u(t).$$

The source term f is of class $L^2(0, \infty; L^2(0, 1))$, hence by maximal regularity [bensoussan'control] we have

$$\xi \in H^1(0, \infty; L^2(0, 1)) \cap L^2(0, \infty; H^2(0, 1)).$$

The function z is therefore also in the above class and we are allowed to pass (31) to the Laplace transform with respect to t : for all $s \in \mathbb{C}_+$, the function $\hat{z}(s, \cdot)$ satisfies

$$\begin{cases} s\hat{z}(s, x) = \partial_{xx}\hat{z}(s, x), & \text{a.e. } x \in (0, 1), \\ \partial_x\hat{z}(s, 1) = \hat{u}(s), \\ \hat{z}(s, 0) = 0, \end{cases}$$

from which we deduce that

$$\hat{z}(s, x) = \frac{\hat{u}(s) \sinh(\sqrt{s}x)}{\sqrt{s} \cosh(\sqrt{s})}.$$

We then evaluate

$$\hat{y}(s) = \partial_x \hat{z}(s, 0) = \frac{\hat{u}(s)}{\cosh \sqrt{s}},$$

so that the transfer function of (31) is thus

$$\mathbf{H}_{\text{Neu} - \text{Neu}}(s) = \frac{1}{\cosh \sqrt{s}}.$$

Notice that the above function is bounded in modulus on $\mathbb{C}_+$, from which one deduces by approximation arguments that for general $u \in L^2(0, \infty)$, the output y is $L^2(0, \infty)$.

Repeating the analysis proposed in Section 2, we deduce that the trackable subspace in infinite time satisfies

$$\mathcal{Y}_{\text{Neu} - \text{Neu}}(0, \infty) = \mathcal{G}_{2,1/\sqrt{2},0}(0, \infty) \cap C_{(0)}^\infty[0, \infty).$$

For the trackable subspace in finite time, we need the reachable space of the control system (31), that is defined by the same equations as (31) but with nonzero initial data disregarding the output y . From [hartmann-orsoni], this reachable space is given by

$$\mathcal{R}_{\text{Neu} - \text{Neu}} = \{f \in \mathcal{H}(\Omega) : f \text{ odd}, \quad f' \in L^2(\Omega)\}.$$

Repeating the proof of Theorem 1.2, one arrives at

$$\mathcal{Y}_{\text{Neu} - \text{Neu}}(0, T) = \left\{ y \in \mathcal{G}_{2,1/\sqrt{2},0}(0, T) \cap C_{(0)}^\infty[0, T] : \sum_{k=0}^{\infty} y^{(k)}(T) \frac{\zeta^{2k}}{(2k)!} \in A^2(\Omega) \right\}.$$

3.1.2. *Dirichlet-to-Neumann.* We consider now the system

$$\begin{cases} z_t(t, x) &= z_{xx}(t, x), \quad 0 < x < 1, \quad t > 0, \\ z(t, 1) &= u(t), \\ z(t, 0) &= 0, \\ z(0, x) &= 0, \\ y(t) &= z_x(t, 0). \end{cases} \quad (32)$$

When the initial data in (32) is nonzero and disregarding the output y , it is a well-posed system on the state space $H^{-1}(0, 1) := (H_0^1(0, 1))'$, with reachable space

$$\mathcal{R}_{\text{Dir} - \text{Neu}} = \{f \in \mathcal{H}(\Omega) : f \text{ odd}, \quad f \in L^2(\Omega)\}.$$

Observe that when the initial data $z^0 \in H^{-1}(0, 1)$ of (32) is nonzero and the control u is zero, the output $y(t) = z_x(t, 0)$ may fail to be integrable around $t = 0$. But this does not affect our method as explained in Remark 2.3. Reasoning as in the previous subsection one sees that the transfer function of (32) is given by

$$\mathbf{H}_{\text{Dir} - \text{Neu}}(s) = \frac{\sqrt{s}}{\sinh \sqrt{s}},$$

so that

$$\mathcal{Y}_{\text{Dir} - \text{Neu}}(0, \infty) = \mathcal{G}_{2,1/\sqrt{2},-1/2}(0, \infty) \cap C_{(0)}^\infty[0, \infty),$$

and

$$\begin{aligned} &\mathcal{Y}_{\text{Dir} - \text{Neu}}(0, T) \\ &= \left\{ y \in \mathcal{G}_{2,1/\sqrt{2},-1/2}(0, T) \cap C_{(0)}^\infty[0, T] : \sum_{k=0}^{\infty} y^{(k)}(T) \frac{\zeta^{2k+1}}{(2k+1)!} \in A^2(\Omega) \right\}. \end{aligned}$$

3.1.3. *Dirichlet-to-Dirichlet.* Finally consider the system

$$\begin{cases} z_t(t, x) &= z_{xx}(t, x), \quad 0 < x < 1, \quad t > 0, \\ z(t, 1) &= u(t), \\ z_x(t, 0) &= 0, \\ z(0, x) &= 0, \\ y(t) &= z(t, 0). \end{cases} \quad (33)$$

When the initial data in (33) is nonzero and disregarding the output y , it is a well-posed system on the state space $H^{-1}(0, 1)$. We have

$$\mathcal{R}_{\text{Dir} - \text{Dir}} = \{f \in \mathcal{H}(\Omega) : f \text{ even}, \quad f \in L^2(\Omega)\}, \quad \mathbf{H}_{\text{Dir} - \text{Dir}}(s) = \frac{1}{\cosh \sqrt{s}},$$

so that

$$\mathcal{Y}_{\text{Dir} - \text{Dir}}(0, \infty) = \mathcal{G}_{2,1/\sqrt{2},0}(0, \infty) \cap C_{(0)}^\infty[0, \infty),$$

and

$$\mathcal{Y}_{\text{Dir} - \text{Dir}}(0, T) = \left\{ y \in \mathcal{G}_{2,1/\sqrt{2},0}(0, T) \cap C_{(0)}^\infty[0, T] : \sum_{k=0}^{\infty} y^{(k)}(T) \frac{\zeta^{2k}}{(2k)!} \in A^2(\Omega) \right\}.$$

3.2. **Smooth control laws.** For a given integer $p \geq 1$, consider the case where the control u belongs to $H_0^p(0, \infty)$, which is defined as

$$H_0^p(0, \infty) := \left\{ u \in H^p(0, \infty) : u(0) = u'(0) = \dots = u^{(p-1)}(0) = 0 \right\}.$$

For such u , the relation (12) can be rewritten using

$$\widehat{u^{(p)}}(s) = s^p \hat{u}(s).$$

This allows us to identify the set $\mathcal{Y}^p(0, \infty)$ of these signals that can be tracked with $H_0^p(0, \infty)$ controls. For finite time considerations, Theorem 1.2 can be adapted using the knowledge we have of the reachable space of the heat equation with smooth inputs, see [hartmann-orsoni].

3.3. **Other point measurement.** For system (1), the output y is the Dirichlet trace at $x = 0$. Consider the same problem but with measurement taken at some $0 < x_0 < 1$:

$$\begin{cases} z_t(t, x) &= z_{xx}(t, x), \quad 0 < x < 1, \quad t > 0, \\ z_x(t, 1) &= u(t), \\ z_x(t, 0) &= 0, \\ z(0, x) &= 0, \\ y(t) &= z(t, x_0). \end{cases} \quad (34)$$

The proof of Theorem 1.1 can be easily adapted to cover this case. Let us denote by $\mathcal{Y}_{x_0}(0, \infty)$ the corresponding trackable subspace.

Theorem 3.1. *For every $0 < x_0 < 1$, the set $\mathcal{Y}_{x_0}(0, \infty)$ consists of those $\varphi \in C_{(0)}^\infty[0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \left(\frac{\|\varphi^{(k+1)}\|_{L^2(0, \infty)}}{(1+k)^{3/4}(2k)!} \left(\frac{1-x_0}{\sqrt{2}} \right)^{2k} \right)^2 < \infty. \quad (35)$$

The Gevrey radius $(1-x_0)/\sqrt{2}$ is the radius of the circle centered at x_0 and inscribed in Ω .

Proof. Let $u \in L^2(0, \infty)$ and denote y the associated output of (34). Reasoning as in the proof of Theorem 1.1 we deduce that y is Laplace transformable, with abscissa of absolute convergence no greater than 0, and satisfies

$$\hat{y}(s) = \frac{\cosh(\sqrt{s}x_0)}{\sqrt{s} \sinh(\sqrt{s})} \hat{u}(s), \quad \operatorname{Re} s > 0.$$

We rewrite the above as

$$\psi(s) \hat{y}(s) = \hat{u}(s), \quad \psi(s) := \frac{\sinh \sqrt{s}}{\sqrt{s} \cosh(\sqrt{s}x_0)}.$$

The function $1/\psi$ is bounded on $\mathbb{C}_+$ from the Phragmén-Lindelöf principle, hence $\hat{y} \in L^2(0, \infty)$. To conclude as in the proof of Theorem 1.1 we wish to apply Proposition 1.6. However, the function ψ is not entire, but holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. Coming back to [foures] we see that actually Proposition 1.6 holds under the mere assumption: $\phi \in \mathcal{H}(\mathbb{C}_+) \cap C^0(\mathbb{C}_+ \cup i\mathbb{R})$ has exponential order < 1 on $\mathbb{C}_+$. The proof is thus the same as for Theorem 1.1. $\square$

In view of Theorem 1.2 one may hope for a characterization of $\mathcal{Y}_{x_0}(0, T)$ when $0 < T < \infty$. However, the flat-output method brings

$$z(t, x) = \sum_{k=0}^{\infty} y^{(k)}(t) \frac{(x - x_0)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \partial_t^k z_x(t, x_0) \frac{(x - x_0)^{2k+1}}{(2k+1)!},$$

hence our method does not allow us to characterize $\mathcal{Y}_{x_0}(0, T)$. This suggests the investigation of the following system

$$\begin{cases} z_t(t, x) = z_{xx}(t, x), & -1 < x < 1, \quad t > 0, \\ z(t, -1) = u_1(t), \\ z(t, 1) = u_2(t), \\ z(0, x) = 0, \\ y_1(t) = z(t, x_0), \\ y_2(t) = z_x(t, x_0). \end{cases} \quad (36)$$

For brevity we leave open the determination of the trackable subspace for system (36).

4. Interpolation in Gevrey classes of order 2. In this Section we prove Proposition 1.9 and Corollary 1.10.

Proof of Proposition 1.9. We begin by proving the result in case $R = 1/\sqrt{2}$ and $\gamma = 0$. This case is special because the system “Dirichlet-to-Dirichlet” (33) has its trackable subspace satisfying

$$\mathcal{Y}_{\text{Dir} - \text{Dir}}(0, \infty) = \mathcal{G}_{2, 1/\sqrt{2}, 0}(0, \infty) \cap C_{(0)}^\infty[0, \infty).$$

Moreover, its reachable space is given by

$$\mathcal{R}_{\text{Dir} - \text{Dir}} = \{f \in \mathcal{H}(\Omega) : f \text{ even}, \quad f \in L^2(\Omega)\}. \quad (37)$$

Let us now show that

$$\mathcal{B}_{t_0} \mathcal{G}_{2, 1/\sqrt{2}, 0}(I) = \left\{ (a_k) : \sum_{k=0}^{\infty} a_k \frac{\zeta^{2k}}{(2k)!} \in A^2(\Omega) \right\}. \quad (38)$$

By cutoff arguments, there is no loss of generality in assuming that I is bounded (see [reachable martin]). By translation we may further assume that $\inf I = 0$.

Then, on the one hand, take (a_k) in $\mathcal{B}_{t_0}\mathcal{G}_{2,1/\sqrt{2},0}(I)$ and fix a preimage φ . Owing to Corollary 1.3 the function φ is in $\mathcal{Y}_{\text{Dir}-\text{Dir}}(0, t_0)$ and thus satisfies

$$\sum_{k=0}^{\infty} a_k \frac{\zeta^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \varphi^{(k)}(t_0) \frac{\zeta^{2k}}{(2k)!} \in \mathcal{R}_{\text{Dir}-\text{Dir}}. \quad (39)$$

This shows the direct inclusion of (38). On the other hand, let (a_k) be a sequence such that

$$\sum_{k=0}^{\infty} a_k \frac{\zeta^{2k}}{(2k)!} \in A^2(\Omega). \quad (40)$$

By definition of the reachable space and (37), there is $u \in L^2(0, t_0)$ such that the solution z of (33) with control u reaches the state defined by the power series in (40), in time t_0 . We extend the control u by 0 on (t_0, ∞) , which extends z for all positive times, which shows that $(a_k) \in \mathcal{B}_{t_0}\mathcal{G}_{2,1/\sqrt{2},0}(I)$. This completes the proof of (38).

Acting on (a_k) by scaling and shift, we deduce that

$$\mathcal{B}_{t_0}\mathcal{G}_{2,R,p}(I) = \left\{ (a_k) : \sum_{k=\max(0,-p)}^{\infty} a_{k+p} \frac{(\sqrt{2}R\zeta)^{2k}}{(2k)!} \in A^2(\Omega) \right\}, \quad R > 0, \quad p \in \mathbb{Z}.$$

To deal with the case $\gamma = p - 1/2$, $p \in \mathbb{Z}$, we repeat the same analysis for system (32). We obtain

$$\mathcal{B}_{t_0}\mathcal{G}_{2,1/\sqrt{2},-1/2}(I) = \mathcal{B}_{t_0}\mathcal{Y}_{\text{Dir}-\text{Neu}}(0, \infty) = \left\{ (a_k) : \sum_{k=0}^{\infty} a_k \frac{\zeta^{2k+1}}{(2k+1)!} \in A^2(\Omega) \right\},$$

and

$$\mathcal{B}_{t_0}\mathcal{G}_{2,R,p-1/2}(I) = \left\{ (a_k) : \sum_{k=\max(0,-p)}^{\infty} a_{k+p} \frac{(\sqrt{2}R\zeta)^{2k+1}}{(2k+1)!} \in A^2(\Omega) \right\},$$

which completes the proof of Proposition 1.9. $\square$

Proof of Corollary 1.10. We fix (a_n) an arbitrary sequence of complex numbers and we introduce the quantity

$$\mathfrak{R}_a := \sup\{R > 0 : \exists \varphi \in \mathcal{G}_{2,R,0}(-1, 1), \quad \forall n \in \mathbb{N}, \quad \varphi^{(n)}(0) = a_n\}.$$

From Proposition 1.9 we clearly have

$$\mathfrak{R}_a = \sup \left\{ R \geq 0 : \sum_{k=0}^{\infty} a_k \frac{(\sqrt{2}R\zeta)^{2k}}{(2k)!} \in A^2(\Omega) \right\},$$

hence we are left to show that $\mathfrak{R}_a = R_a$. On the one hand, for all $0 < R' < R$ we have $G^{2,R}[-1, 1] \subset \mathcal{G}_{2,R',0}(-1, 1)$, hence $R_a \leq \mathfrak{R}_a$. On the other hand, let $0 < R < \mathfrak{R}_a$ and $\varphi \in \mathcal{G}_{2,R,0}(-1, 1)$ interpolating (a_n) . From [reachable·martin] we can further assume that φ is flat at ± 1 , so that in particular the Poincaré inequality applies to φ and all its successive derivatives. We easily deduce that for all $0 < R' < R$ there holds $\varphi \in G^{2,R'}[-1, 1]$, hence $R' \leq R_a$. We deduce that $\mathfrak{R}_a \leq R_a$, which concludes the proof. $\square$

In Proposition 1.4 we have given an example of a sequence (a_k) whose Gevrey radius is 1, but which cannot be interpolated by a function that is Gevrey of radius $1/\sqrt{2}$. Corollary 1.10 allows us to obtain the other extreme case: the sequence

$$a_k := \frac{(2k)!2^k i^k}{\sqrt{k}}, \quad a_0 := 0,$$

can be interpolated by a function which is Gevrey of order 2 and radius $1/\sqrt{2} - \epsilon$, for all $\epsilon > 0$ (see (30)).

Appendix A. More on the interpolation problem. In this Appendix, we provide a detailed analysis of the interpolation problem in Gevrey classes of order $s > 1$, by completing the proof of Theorems 1 and 1a of [mitjagin], which are only sketched in the cited paper.

Let us first clarify Remark 1.5: in [reachable·martin], it is claimed that for every sequence (a_n) such that

$$|a_n| \lesssim \frac{(2n)!}{R_0^{2n}},$$

and for every $\epsilon > 0$ with $\epsilon < R_0$, the sequence (a_n) can be interpolated by a function $\varphi \in C^\infty(\mathbb{R})$ such that

$$|\varphi^{(n)}(t)| \lesssim \frac{(2n)!}{(\rho^{MRR} R_0 - \epsilon)^{2n}},$$

where

$$\rho^{MRR} := \exp\left(-\frac{1}{2e}\right) \approx 0.832 > 0.707 \approx \frac{1}{\sqrt{2}} = \rho_2.$$

This contradicts Proposition 1.4 and [mitjagin], which impose the loss factor ρ^{MRR} to be $\leq 1/\sqrt{2}$.

We believe that, unfortunately, the proof of [reachable·martin] is flawed. Indeed, to go further into the details of the proof in [reachable·martin], the essence of their Theorem 3.2 lies in their Proposition 3.6. To prove this last Proposition, the authors build $f(x) = \sum_p d_p \varphi_p(x) x^p / p!$ where each φ_p is built according to their Corollary 3.5. The constant C appearing in their Corollary 3.5 implicitly depends on δ and on the sequence (a_k) . However, in the proof of their Proposition 3.6, this dependency is not taken into account, and the constant C appearing in the estimate of $\varphi_p^{(j)}(x)$ page 195 *a priori* depends at least on p , which prevents one from concluding as the authors claim.

Note further that [reachable·martin] may be used to show that the interpolation problem can be solved in Gevrey classes of order $s > 1$, with loss factor

$$\rho_s^{MRR} := \exp\left(-\frac{1}{es}\right).$$

As explained above, we believe that the proof of [reachable·martin] is flawed. In fact, for $1 < s < 3$ we have $\rho_s^{MRR} > \rho_s$, hence a contradiction with Mitjagin's results. For $s > 4$ we have $\rho_s^{MRR} < \rho_s$, hence ρ_s^{MRR} is a valid loss factor, although sub-optimal.

We believe this is a minor inaccuracy. Indeed, the result itself of [reachable·martin] might be corrected to yield a valid result, by considering the loss factor of [mitjagin], whereas the proof of [reachable·martin] might be corrected to yield a valid result, probably with a suboptimal loss factor, see [petzsche]. As far as we are concerned,

whenever Theorem 3.2 or Proposition 3.6 from [reachable·martin] is used, the resulting assertion can be corrected by merely modifying a numerical constant. Moreover, this does not detract from the importance of the article [reachable·martin], which is the first to successfully describe the reachable states of the heat equation using holomorphic functions.

A.1. Generalities. In this subsection we study the interpolation problem in a rather general setting. In order to remain close to Mitjagin's notation, we change our convention for Gevrey functions. This is a minor modification and we shall explain how to relate the loss factors between the two conventions.

Definition A.1. Let

$$-\infty < a < b < \infty, \quad c \in [a, b], \quad A > 0, \quad 1 \leq \Gamma < \infty, \quad 1 < s < \infty.$$

We say that the interpolation problem associated with (a, b, c, A, s) is solvable with loss Γ if, for every sequence of complex numbers (a_n) such that

$$\exists C > 0, \quad \forall n \in \mathbb{N}, \quad |a_n| \leq CA^n n^{ns},$$

and for every $\epsilon > 0$, there exists a function $\varphi \in C^\infty[a, b]$ such that

$$\forall n \in \mathbb{N}, \quad \varphi^{(n)}(c) = a_n,$$

and

$$\exists C' > 0, \quad \forall n \in \mathbb{N}, \quad \forall t \in [a, b], \quad |\varphi^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns}.$$

If Γ_s stands for the optimal loss factor for the above definition and ρ_s stands for the optimal loss factor with the conventions used in (16) and (17), an application of Stirling's formula yields $\rho_s = \Gamma_s^{-1/s}$.

Definition A.1 *a priori* depends on the parameters (a, b, c, A, s) . We claim that actually it does not depend on a, b, c, A when s is fixed, in the sense that changing a, b, c and A does not affect the loss factor Γ .

Proposition A.2. Fix $s > 1$ and assume that the interpolation problem associated with (a, b, c, A, s) is solvable with loss Γ . Then, the interpolation problem associated with (a', b', c', A', s) is solvable with the same loss factor Γ .

Proof. Assume that the interpolation problem associated with (a, b, c, A, s) is solvable with loss factor Γ . We show that for every (a', b', c', A', s) as above, the associated problem is solvable with loss Γ . Clearly, for any translation $\tau \in \mathbb{R}$, the problem associated with $(a + \tau, b + \tau, c + \tau, A, s)$ is solvable with the same loss factor. We may therefore assume that $a = a' = 0$. Let us then use a dilatation: if $\varphi \in C^\infty[0, b]$ is such that

$$|\varphi^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns},$$

then for $\lambda > 0$ we consider $\varphi_\lambda(t) := \varphi(\lambda t)$, which is such that

$$\varphi_\lambda \in C^\infty[0, b/\lambda], \quad \varphi_\lambda^{(n)}(t) = \lambda^n \varphi^{(n)}(t), \quad |\varphi_\lambda^{(n)}(t)| \leq C'(\Gamma A \lambda + \epsilon \lambda)^n n^{ns}.$$

Thus the problem of parameters $(0, b/\lambda, c/\lambda, A\lambda, s)$ is solved with loss Γ . We can therefore assume that $b = b' = 1$.

We then claim that if the problem of parameters $(0, 1, c, A, s)$ is solvable with loss Γ for $0 < c < 1$, then for every $c' \in (0, 1)$ the problem of parameters $(0, 1, c', A, s)$ is solvable with loss Γ . Indeed, let $\epsilon > 0$ and (a_n) be any sequence such that

$$|a_n| \leq CA^n n^{ns}.$$

We invoke the solvability of the problem associated with $(0, 1, c, A, s)$ to take $\varphi \in C^\infty[0, 1]$ such that

$$\varphi^{(n)}(c) = a_n, \quad |\varphi^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns}.$$

We then consider

$$\psi(t) := \varphi(t - c' + c)\chi(t),$$

where $\chi \in C^\infty(\mathbb{R})$ is Gevrey of order $r \in (1, s)$, is constant to 1 on a small neighborhood of c' , and flat at $c' - c$ and $1 + c' - c$. Since $1 < r < s$, the constructed function ψ has the same Gevrey regularity as φ in the sense that

$$|\psi^{(n)}(t)| \leq C''(\Gamma A + \epsilon)^n n^{ns},$$

(see **[reachable martin]**). Moreover, since χ is 0 outside of $(c' - c, 1 + c' - c)$, ψ is well-defined on $\mathbb{R}$ (so it is also defined on $[0, 1]$), and has the same derivatives at c' as φ has at c . Thus the problem of parameters $(0, 1, c', A, s)$ is solvable, with the same loss Γ .

Next we show that the cases $c = 0$ or $c = 1$ are equivalent. Assume for instance that the problem of parameters $(0, 1, 0, A, s)$ is solvable with loss Γ . For any sequence (a_n) such that

$$|a_n| \leq CA^n n^{ns},$$

we invoke its solution for the sequence $((-1)^n a_n)$ to obtain $\varphi \in C^\infty[0, 1]$ such that

$$\varphi^{(n)}(0) = (-1)^n a_n, \quad |\varphi^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns}.$$

We then put $\psi(t) := \varphi(1 - t)$, which has the same Gevrey regularity as φ and is such that

$$\psi^{(n)}(1) = (-1)^n \varphi^{(n)}(0) = a_n.$$

The converse implication is proved in the same way, so that $c = 0$ and $c = 1$ are equivalent.

Assume now that the problem of parameters $(0, 1, 0, A, s)$ is solvable with loss Γ . We show that, for every $0 < c < 1$ the problem of parameters $(0, 1, c, A, s)$ is solvable with loss Γ . In fact we will show that $(-1, 1, 0, A, s)$ is solvable with loss Γ , which is easily seen as a stronger statement, by translation and dilatation arguments. We let (a_n) be a sequence with

$$|a_n| \leq CA^n n^{ns}.$$

We introduce the “odd” and “even” sequences

$$a_n^o = \begin{cases} a_{2k+1}, & n = 2k+1, \\ 0, & n = 2k, \end{cases}, \quad a_n^e = \begin{cases} a_{2k}, & n = 2k, \\ 0, & n = 2k+1, \end{cases}$$

and take $\varphi, \psi \in C^\infty[0, 1]$ so that

$$|\varphi^{(n)}(t)|, |\psi^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns}, \quad \varphi^{(n)}(0) = a_n^o, \quad \psi^{(n)}(0) = a_n^e.$$

We then consider $\tilde{\varphi}$ the odd extension of φ and $\tilde{\psi}$ the even extension of ψ . Clearly, they are both C^∞ on $[-1, 0]$ and $[0, 1]$, and on each of these two intervals they satisfy the Gevrey estimate

$$|\tilde{\psi}^{(n)}(t)|, |\tilde{\varphi}^{(n)}(t)| \leq C'(\Gamma A + \epsilon)^n n^{ns}.$$

To check that $\tilde{\varphi}$ and $\tilde{\psi}$ satisfy the required Gevrey estimate on $[-1, 1]$, it is thus enough to check that their derivatives at $0^\pm$ both match. We have

$$\tilde{\varphi}^{(n)}(0^+) = \varphi^{(n)}(0) = a_n^o, \quad \tilde{\varphi}^{(n)}(0^-) = (-1)^{n+1} \varphi^{(n)}(0) = (-1)^{n+1} a_n^o,$$

and these two values agree. Similarly,

$$\tilde{\psi}^{(n)}(0^+) = \psi^{(n)}(0) = a_n^e, \quad \tilde{\psi}^{(n)}(0^-) = (-1)^n \psi^{(n)}(0) = (-1)^n a_n^e,$$

so that $\phi := \tilde{\varphi} + \tilde{\psi}$ solves the problem.

We can thus put $c = c' = 0$, hence we are left to show that if the problem $(0, 1, 0, A, s)$ is solvable with loss Γ , then, for every $A' > 0$, the problem $(0, 1, 0, A', s)$ is solvable with loss Γ . Assume that the problem associated with $(0, 1, 0, A, s)$ is solvable with loss Γ , using the dilatation φ_λ we see that for every $\lambda > 0$, the problem associated with $(0, 1/\lambda, 0, A\lambda, s)$ is solvable with loss Γ . We distinguish two cases: if $\lambda \leq 1$, then the problem associated with $(0, 1, 0, A\lambda, s)$ is trivially solvable. If on the contrary $\lambda > 1$, using a cutoff that is Gevrey of order $1 < r < s$, constant to 1 near 0 and flat at $1/\lambda$, we see that the problem associated with $(0, 1, 0, A\lambda, s)$ is also solvable with loss Γ , as in the case of the equivalence for the parameter c . Since $\lambda > 0$ is arbitrary, this finishes the proof. $\square$

Therefore, the only relevant numbers are Γ and s .

A.2. A proof of Mitjagin's Theorem. We now provide a detailed proof of Theorems 1 and 1a of [mitjagin]. Recall the notation

$$\Gamma_\beta := \cos^{-\beta} \frac{\pi}{2\beta}, \quad \beta > 1.$$

We begin by showing that Γ_β is an admissible loss factor.

Theorem A.3. *Let $\beta > 1$ and $B, \epsilon > 0$ be fixed. Then for every sequence (a_n) such that*

$$\exists C > 0, \quad \forall n \in \mathbb{N}, \quad |a_n| \leq CB^n n^{n\beta},$$

there exists $\varphi \in C^\infty[-1, 1]$ such that

$$\exists C' > 0, \quad \forall t \in [-1, 1], \quad \forall n \in \mathbb{N}, \quad |\varphi^{(n)}(t)| \leq C'(\Gamma_\beta B + \epsilon)^n n^{n\beta}$$

together with

$$\forall n \in \mathbb{N}, \quad \varphi^{(n)}(0) = a_n.$$

In order to show this theorem we need some preparation. For given parameters $B > 0$ and $\beta > 1$ we define $C_{\beta B}^\infty$ as the set of those $\phi \in C^\infty[-1, 1]$ such that

$$\|\phi\|_{C_{\beta B}^\infty} := \sup_{n \in \mathbb{N}} \sup_{t \in [-1, 1]} \frac{|\phi^{(n)}(t)|}{B^n n^{n\beta}} < \infty.$$

This defines a Banach space.

A functional f on $C_{\beta B}^\infty$ is said to be concentrated (at 0) if it vanishes on any $\phi \in C_{\beta B}^\infty$ that is equal to 0 around $t = 0$. For such a functional f we construct an entire series, roughly speaking the Fourier transform of f , in the following Lemma. We introduce a function $h \in C^\infty(\mathbb{R})$ such that

$$|h^{(n)}(t)| \leq CA^n n^{n\gamma}, \quad h(t) = 1 \quad (|t| < 1/2), \quad h(t) = 0 \quad (|t| > 1),$$

for some constants $1 < \gamma < \beta$ and $A, C > 0$.

Lemma A.4. *Let f be a concentrated functional on $C_{\beta B}^\infty$, $\epsilon > 0$ and h as above. We then have*

1. For every $s \in \mathbb{C}$, the function

$$\phi_s(t) := e^{ist} h(t/\epsilon),$$

is in $C_{\beta B}^\infty$, and verifies the estimate: for any $\delta > 0$, we have

$$\|\phi_s\|_{C_{\beta,B}^\infty} \leq C_1 \exp \left[\epsilon |\tau| + \frac{\beta}{e} \left(\frac{1+\delta}{B} r \right)^{1/\beta} \right], \quad (41)$$

where $r = |s|$, $s = \xi + i\tau$, and $C_1 = C_1(C, A, \epsilon, \delta, \beta, B)$.

2. The function $\Phi(s) := \langle f, \phi_s \rangle$ is well-defined, entire on $\mathbb{C}$, and does not depend on $\epsilon > 0$ or h as above.

3. The function Φ satisfies the estimate

$$|\Phi(s)| \leq \|f\| C_1 \exp \left(\frac{\beta}{e} \left(\frac{1+\delta}{B} r \right)^{1/\beta} \frac{1}{\cos(\pi/(2\beta))} \right). \quad (42)$$

4. We have the estimation of the Taylor coefficients of Φ

$$|f_n| \leq C_1 \|f\| \left(\frac{1+\delta}{B \cos^\beta \frac{\pi}{2\beta}} \right)^n n^{-n\beta}, \quad \Phi(s) = \sum_{n=0}^{\infty} f_n s^n. \quad (43)$$

Proof. 1. The function ϕ_s is clearly in $C^\infty[-1, 1]$, the goal is thus to estimate its derivatives. For this we use the Leibniz formula

$$\phi_s^{(n)}(t) = \sum_{k=0}^n \binom{n}{k} (is)^k e^{ist} \frac{1}{\epsilon^{n-k}} h^{(n-k)} \left(\frac{t}{\epsilon} \right).$$

Since $h \equiv 0$ outside $[-1, 1]$, we deduce that

$$\left| e^{ist} h^{(p)} \left(\frac{t}{\epsilon} \right) \right| \leq e^{\epsilon |\tau|} C A^p p^{\gamma p}, \quad p \in \mathbb{N}.$$

We deduce that

$$\begin{aligned} |\phi_s^{(n)}(t)| &\leq \sum_{k=0}^n \binom{n}{k} r^k e^{\epsilon |\tau|} C \left(\frac{A}{\epsilon} \right)^{n-k} (n-k)^{(n-k)\gamma} \\ &\leq C e^{\epsilon |\tau|} \sum_{k=0}^n \binom{n}{k} r^k \left(\frac{A}{\epsilon} \right)^{n-k} n^{(n-k)\gamma} \\ &= C e^{\epsilon |\tau|} \left(r + \frac{An^\gamma}{\epsilon} \right)^n, \end{aligned}$$

where we recall that $r = |s|$. In view of the definition of the $\|\cdot\|_{C_{\beta B}^\infty}$ -norm, the estimate (41) thus boils down to proving that

$$\forall \delta > 0, \quad \exists C_1 > 0, \quad \forall n \in \mathbb{N}, \quad \forall r > 0,$$

$$\left(r + \frac{An^\gamma}{\epsilon} \right)^n \leq C_1 B^n n^{n\beta} \exp \left[\frac{\beta}{e} \left(\frac{1+\delta}{B} r \right)^{1/\beta} \right].$$

If $n = 0$ the estimate is obvious (put $C_1 = 1$), hence we may take $n \geq 1$ without loss of generality. We will obtain the above estimate by considering separately three ranges for the parameters r and n . We introduce another

parameter $\gamma < \lambda < \beta$, and the first range we consider is $r \leq n^\lambda$. For such r and n , we have

$$\left(r + \frac{An^\gamma}{\epsilon}\right)^n \leq \left(n^\lambda + \frac{An^\gamma}{\epsilon}\right)^n = n^{\lambda n} \left(1 + \frac{An^{\gamma-\lambda}}{\epsilon}\right)^n \leq n^{\lambda n} e^{C(\epsilon, A, \gamma, \lambda) n^{1+\gamma-\lambda}},$$

where $C(\epsilon, A, \gamma, \lambda)$ is a constant only depending on $\epsilon, A, \gamma, \lambda$. Because $\lambda < \beta$, the term $n^{\lambda n} e^{C(\epsilon, A, \gamma, \lambda) n^{1+\gamma-\lambda}}$ is negligible compared to $B^n n^{n\beta}$ as $n \rightarrow \infty$, hence

$$\left(r + \frac{An^\gamma}{\epsilon}\right)^n \leq C(\epsilon, A, B, \gamma, \lambda, \beta) B^n n^{n\beta}, \quad r \leq n^\lambda.$$

We then assume that $r \geq n^\lambda$, and compute

$$\begin{aligned} \left(r + \frac{An^\gamma}{\epsilon}\right)^n &= \left(r + \frac{A(n^\lambda)^{\gamma/\lambda}}{\epsilon}\right)^n \\ &\leq \left(r + \frac{Ar^{\gamma/\lambda}}{\epsilon}\right)^n \\ &= r^n \left(1 + \frac{Ar^{\frac{\gamma}{\lambda}-1}}{\epsilon}\right)^n. \end{aligned}$$

Because $0 < \gamma/\lambda < 1$, we have

$$\forall \delta > 0, \quad \exists R = R(A, \epsilon, \delta, \gamma/\lambda) > 0, \quad \forall r > R, \quad 1 + \frac{Ar^{\frac{\gamma}{\lambda}-1}}{\epsilon} \leq 1 + \delta.$$

Fix $\delta > 0$ arbitrary and introduce an $R > 0$ as above. The second regime we consider is $r \geq \max(n^\lambda, R)$, for which we find

$$\left(r + \frac{An^\gamma}{\epsilon}\right)^n \leq r^n (1 + \delta)^n.$$

We then fix $r \geq \max(n^\lambda, R)$ and compute

$$\sup_{x>0} \frac{r^x (1 + \delta)^x}{B^x x^{x\beta}} = \sup_{y>0} \frac{D^y}{y^y}, \quad D := \left(\frac{r(1 + \delta)}{B}\right)^{1/\beta} \beta, \quad y = x\beta.$$

The above supremum is reached at $y = D/e$, for which

$$\sup_{x>0} \frac{r^x (1 + \delta)^x}{B^x x^{x\beta}} = e^{D/e} = \exp \left[\frac{1}{e} \left(\frac{r(1 + \delta)}{B}\right)^{1/\beta} \beta \right].$$

Thus for $n \geq 1$ and $r \geq \max(n^\lambda, R)$ we find

$$\left(r + \frac{An^\gamma}{\epsilon}\right)^n \leq \exp \left[\frac{1}{e} \left(\frac{r(1 + \delta)}{B}\right)^{1/\beta} \beta \right] B^n n^{n\beta}.$$

The third and last regime we investigate is when $n \geq 1$ and $n^\lambda \leq r \leq \max(R, n^\lambda)$. If $R < n^\lambda$ we are back to the first regime, hence we are left with $n^\lambda \leq R$. There are only finitely many such values of n , for which

$$\left(r + \frac{An^\gamma}{\epsilon}\right)^n \leq C(A, \epsilon, \lambda, \gamma, R) = C(A, \epsilon, \lambda, \gamma, \delta).$$

2. The fact that Φ is well-defined as a function $\mathbb{C} \rightarrow \mathbb{C}$ follows from the previous point. It is elementary to check that it is holomorphic on $\mathbb{C}$. Because f is concentrated, the function Φ does not depend on h or ϵ .

3. In order to show (42), we start from (41) and deduce

$$|\Phi(s)| = |\langle f, \phi_s \rangle| \leq \|f\| \|\phi_s\|_{C_{\beta, B}^\infty} \leq \|f\| C_1 \exp \left[\epsilon |\tau| + \frac{\beta}{e} \left(\frac{1+\delta}{B} r \right)^{1/\beta} \right]$$

where, as recalled, we constantly use the notations $r = |s|$, $s = \xi + i\tau$. In particular, the function Φ satisfies

$$\forall \epsilon > 0, \quad \exists C_\epsilon > 0, \quad \forall s \in \mathbb{C}, \quad |\Phi(s)| \leq C_\epsilon e^{\epsilon r}, \quad (44)$$

i.e. it is of order 1 and of type 0 on $\mathbb{C}$. Note however that on the real axis we have a better estimate, of the form

$$\exists C, K > 0, \quad \forall \xi \in \mathbb{R}, \quad |\Phi(\xi)| \leq C e^{K|\xi|^{1/\beta}}, \quad (45)$$

i.e. Φ is of finite type for the order $\leq 1/\beta$ on $\mathbb{R}$. We then use the following version of the Phragmén-Lindelöf's method.

Theorem A.5. *Let $0 < \rho < 1$ and $M, K \geq 0$. Assume that $g \in \mathcal{H}(\mathbb{C}_+) \cap C^0(\mathbb{C}_+ \cup i\mathbb{R})$ is of order 1 with minimal type 0, together with*

$$\forall y \in \mathbb{R}, \quad |g(iy)| \leq M e^{K|y|^\rho}.$$

Then, for every $z \in \mathbb{C}_+$ we have

$$|g(z)| \leq M \exp \left(\frac{K|z|^\rho}{\cos(\pi\rho/2)} \right).$$

Let us show this version of the Phragmén-Lindelöf principle. We will use the following “standard” version of the Phragmén-Lindelöf principle, see [K1].

Theorem A.6. *Let $\alpha \geq 1/2$, $\Omega = \{re^{i\theta} : r > 0, \quad |\theta| < \frac{\pi}{2\alpha}\}$, and $g \in \mathcal{H}(\Omega) \cap C^0(\overline{\Omega})$ of order $\rho < \alpha$, such that*

$$|g(re^{\pm i\frac{\pi}{2\alpha}})| \leq M.$$

Then, for every $z \in \Omega$ there holds

$$|g(z)| \leq M.$$

To show the less standard version of the Phragmén-Lindelöf's method we fix a function g as in the hypothesis. For every $\kappa > 0$ we consider the function

$$g_\kappa(z) = \exp \left(-\frac{K}{\cos(\pi\rho/2)} z^\rho - \kappa z \right) g(z) \in \mathcal{H}(\mathbb{C}_+) \cap C^0(\mathbb{C}_+ \cup i\mathbb{R}).$$

We apply the “standard” Phragmén-Lindelöf's method quoted above to the function g_κ on the domain $\Omega := \{x + iy : x, y > 0\}$. Note that the quoted result is invariant by rotation, hence we may see Ω as a sector with angle $\pi/2$, and the parameter α is set to 2. On Ω the function g_κ is of order 1, which is strictly less than α . Moreover, on the boundary of Ω lying on the imaginary axis there holds

$$|g_\kappa(iy)| = \exp \left(-\frac{K}{\cos(\pi\rho/2)} \operatorname{Re}((iy)^\rho) \right) |g(iy)| \leq e^{-Ky^\rho} M e^{Ky^\rho} = M.$$

On the boundary of Ω lying on the real axis we proceed as follows: the function g is of order 1 and of type 0, hence

$$\forall \eta > 0, \quad \exists C_\eta > 0, \quad \forall z \in \mathbb{C}_+, \quad |g(z)| \leq C_\eta e^{\eta|z|}.$$

We take $\eta = \kappa$, so that for every $x > 0$,

$$|g_\kappa(x)| \leq \exp\left(-\frac{K}{\cos(\pi\rho/2)}x^\rho - \kappa x\right) C_\kappa e^{\kappa x} \leq C_\kappa.$$

The function $|g_\kappa|$ is thus bounded above on Ω by the constant $\max(M, C_\kappa)$, and the quoted Phragmén-Lindelöf's method yields that

$$\forall z \in \Omega, \quad |g_\kappa(z)| \leq \max(M, C_\kappa).$$

The same reasoning can be applied on the domain $\{x + iy : y < 0 < x\}$, so that

$$\forall z \in \mathbb{C}_+, \quad |g_\kappa(z)| \leq \max(M, C_\kappa).$$

The function $|g_\kappa|$ is thus bounded on $\mathbb{C}_+$, hence from the quoted Phragmén-Lindelöf's method (or rather, the extended maximum principle, see *e.g.* [K1]),

$$\sup_{z \in \mathbb{C}_+} |g_\kappa(z)| \leq \sup_{z \in \partial\mathbb{C}_+} |g_\kappa(z)| \leq M.$$

Now fix $z \in \mathbb{C}_+$, for every $\kappa > 0$ we have $|g_\kappa(z)| \leq M$ and

$$g_\kappa(z) \rightarrow \exp\left(-\frac{K}{\cos(\pi\rho/2)}z^\rho\right) g(z), \quad \kappa \rightarrow 0^+.$$

Thus

$$\exp\left(-\frac{K}{\cos(\pi\rho/2)}\operatorname{Re}(z^\rho)\right) |g(z)| \leq M,$$

which shows the claimed version of the Phragmén-Lindelöf principle. This shows (42).

4. We apply the Cauchy inequality on a disc of radius $0 < r < \infty$ to obtain

$$|f_n| = \left| \frac{\Phi^{(n)}(0)}{n!} \right| \leq \frac{1}{r^n} \max_{|s|=r} |\Phi(s)| \leq \frac{1}{r^n} \|f\| C_1 \exp\left[\frac{\beta}{e \cos(\pi/2\beta)} \left(\frac{1+\delta}{B} r \right)^{1/\beta} \right],$$

where the second inequality is due to (42). Put

$$\mu := \frac{\beta}{e \cos(\pi/2\beta)} \left(\frac{1+\delta}{B} \right)^{1/\beta} > 0,$$

so that the previous estimate reads

$$|f_n| \leq \|f\| C_1 \frac{e^{\mu r^{1/\beta}}}{r^n}.$$

When $n \geq 1$, the above right-hand side reaches its minimal value with respect to r when

$$r = \left(\frac{n\beta}{\mu} \right)^\beta,$$

at which

$$|f_n| \leq \|f\| C_1 \left(\frac{e\mu}{n\beta} \right)^{n\beta} = C_1 \|f\| \left(\frac{1+\delta}{B \cos^\beta \frac{\pi}{2\beta}} \right)^n n^{-n\beta}.$$

□

We note that Φ does not depend on h or ϵ , hence taking $\epsilon = 2$ brings

$$\Phi(s) = \langle f(t), e^{ist} \rangle.$$

We now make the final preparations to show Theorem A.3. For parameters $\beta > 1$ and $B > 0$ we introduce the Hilbert space $L_{\beta B}^2$ which consists of those $\phi \in C^\infty[-1, 1]$ such that

$$\|\phi\|_{L_{\beta B}^2}^2 := \sum_{n=0}^{\infty} \int_{-1}^{+1} \left| \frac{\phi^{(n)}(t)}{B^n n^{n\beta}} \right|^2 dt < +\infty.$$

It is elementary to check that for every $\epsilon > 0$ we have

$$C_{\beta B}^\infty \subset L_{\beta B}^2 \subset C_{\beta, B+\epsilon}^\infty, \quad (46)$$

where the inclusions are continuous and dense³. We will need a similar construction for our spaces of sequences. For a sequence (w_n) of positive numbers we define $K(w_n)$ as the set of those sequences $a = (a_n)$ such that

$$\|a\|_{K(w_n)} := \sup_{n \in \mathbb{N}} \frac{|a_n|}{w_n} < \infty.$$

This defines a Banach space. For parameters $\beta > 1$ and $B > 0$ we define $\ell_{\beta B}^2$ as the set of those sequences a such that

$$\|a\|_{\ell_{\beta B}^2}^2 := \sum_{n=0}^{\infty} \left| \frac{a_n}{B^n n^{n\beta}} \right|^2 < \infty.$$

This defines a Hilbert space, and for every $\epsilon > 0$ we have the continuous and dense inclusions

$$\ell_{\beta B}^2 \subset K(n^{n\beta} B^n) \subset \ell_{\beta, B+\epsilon}^2. \quad (47)$$

Below is the proof of Theorem A.3, the proof we give is not exactly the one given by Mitjagin. Actually, it is longer, but it will make the proof of the optimality of Γ_β more natural.

Proof of Theorem A.3. Fix $\epsilon, B > 0$ and $\beta > 1$. We consider the Borel operator

$$\mathcal{B}_\infty^\epsilon : C_{\beta, \Gamma_\beta B + \epsilon}^\infty \rightarrow K((\Gamma_\beta B + \epsilon)^n n^{n\beta}), \quad \mathcal{B}_\infty^\epsilon \phi = (\phi^{(n)}(0))_{n=0}^\infty,$$

and the inclusion

$$i_\infty^\epsilon : K(B^n n^{n\beta}) \rightarrow K((\Gamma_\beta B + \epsilon)^n n^{n\beta}).$$

The claim is precisely that

$$\forall \epsilon > 0, \quad \text{Range } i_\infty^\epsilon \subset \text{Range } \mathcal{B}_\infty^\epsilon.$$

We will show this by duality arguments, hence it will be convenient to replace the domains and codomains of $\mathcal{B}_\infty^\epsilon$ and i_∞^ϵ by Hilbert spaces⁴. Consider thus the Borel operator

$$\mathcal{B}_2^\epsilon : L_{\beta, \Gamma_\beta B + \epsilon/2}^2 \rightarrow \ell_{\beta, \Gamma_\beta B + \epsilon}^2,$$

as well as the inclusion

$$i_2^\epsilon : \ell_{\beta B}^2 \rightarrow \ell_{\beta, \Gamma_\beta B + \epsilon}^2.$$

³That the first inclusion is bounded is trivial. That the second inclusion is bounded can be shown using the Sobolev embedding $H^1(-1, 1) \subset C[-1, 1]$. That the inclusions are dense is shown by mollifying with a kernel function lying in $C_{\gamma C}^\infty$ for some $1 < \gamma < \beta$ and $C > 0$.

⁴The Douglas Lemma does not hold on arbitrary non-reflexive Banach spaces, see [embry].

From the inclusions (46) and (47), the operators $\mathcal{B}_2^\epsilon$ and i_2^ϵ are well-defined and bounded. Further, we have

$$\text{Range } i_\infty^\epsilon \subset \text{Range } i_2^\epsilon, \quad \text{Range } \mathcal{B}_2^\epsilon \subset \text{Range } \mathcal{B}_\infty^\epsilon,$$

hence it is sufficient for us to show that

$$\forall \epsilon > 0, \quad \text{Range } i_2^\epsilon \subset \text{Range } \mathcal{B}_2^\epsilon.$$

We now fix $\epsilon > 0$ and aim at showing that

$$\text{Range } i_2^{2\epsilon} \subset \text{Range } \mathcal{B}_2^{2\epsilon},$$

which is enough to conclude. For clarity we put $\mathcal{B} := \mathcal{B}_2^{2\epsilon}$ and $i := i_2^{2\epsilon}$. We identify $Y := \ell_{\beta, \Gamma_\beta B + 2\epsilon}^2$ with its dual space, do the same for $\ell_{\beta B}^2$, and consider the true dual of $X := L_{\beta, \Gamma_\beta B + \epsilon}^2$ for the duality arguments. This means that

$$\mathcal{B} : X \rightarrow Y, \quad i : \ell_{\beta B}^2 \rightarrow Y, \quad \mathcal{B}^* : Y \rightarrow X', \quad i^* : Y \rightarrow \ell_{\beta B}^2.$$

From Douglas's lemma [douglass], the range inclusion

$$\text{Range } i \subset \text{Range } \mathcal{B},$$

is equivalent to

$$\exists c > 0, \quad \forall a \in Y, \quad \|\mathcal{B}^* a\|_{X'} \geq c \|i^* a\|_{\ell_{\beta B}^2}, \quad (48)$$

which is what we aim at showing until the end of the proof. From the density of the set of finitely supported sequences in Y , which we denote by $\mathbb{C}^{(\mathbb{N})}$, it is enough to show (48) when $a \in \mathbb{C}^{(\mathbb{N})}$. For such an a , consider the concentrated functional

$$f := \sum_{n=0}^{\infty} a_n \delta^{(n)}, \quad \langle \delta^{(n)}, \phi \rangle := \phi^{(n)}(0).$$

Clearly, $f \in X'$, and we observe that for any $\phi \in X$

$$\begin{aligned} \langle f, \phi \rangle_{X', X} &= \sum_{n=0}^{\infty} a_n \phi^{(n)}(0) \\ &= \sum_{n=0}^{\infty} \frac{[(\Gamma_\beta B + 2\epsilon)^n n^{n\beta}]^2 a_n \cdot (\mathcal{B}\phi)_n}{[(\Gamma_\beta B + 2\epsilon)^n n^{n\beta}]^2} \\ &= \langle \tilde{a}, \mathcal{B}\phi \rangle_Y \\ &= \langle \mathcal{B}^* \tilde{a}, \phi \rangle_{X', X}, \end{aligned}$$

where

$$\tilde{a}_n := [(\Gamma_\beta B + 2\epsilon)^n n^{n\beta}]^2 a_n.$$

In particular, $f = \mathcal{B}^* \tilde{a}$. From

$$\forall b \in Y, \quad \forall n \in \mathbb{N}, \quad (i^* b)_n = \left(\frac{B}{\Gamma_\beta B + 2\epsilon} \right)^{2n} b_n,$$

we see that (48) is equivalent to

$$\sum_{n=0}^{\infty} |B^n n^{n\beta} a_n|^2 \leq c \|f\|_{X'}^2,$$

with a constant $c > 0$ not depending on a . We then invoke Lemma A.4 for the concentrated functional f . To this end, observe that

$$C_{\beta, \Gamma_\beta B + \epsilon/2}^\infty \subset L_{\beta, \Gamma_\beta B + \epsilon}^2,$$

with continuous dense inclusion. Hence, by transposition, we obtain

$$f \in X' \hookrightarrow (C_{\beta, \Gamma_\beta B + \epsilon/2}^\infty)',$$

so that Lemma A.4 applies to f . Plain computations show that the function Φ constructed in Lemma A.4 satisfies

$$\Phi(s) = \sum_{n=0}^{\infty} a_n (is)^n,$$

so that the estimate (43) brings

$$|a_n| \leq C \|f\|_{X'} \left(\frac{1+\delta}{\Gamma_\beta B + \epsilon/2} \Gamma_\beta \right)^n n^{-n\beta},$$

for a constant $C = C(\delta, \beta, B, \epsilon)$, where $\delta > 0$ is to be tuned later on. We deduce

$$|B^n n^{n\beta} a_n| \leq C \|f\|_{X'} \left(\frac{1+\delta}{1 + \frac{\epsilon}{2\Gamma_\beta B}} \right)^n,$$

and observe that as β, B, ϵ are fixed, we can take $\delta > 0$ small enough so that the right-hand side is summable. $\square$

Remark A.7. Douglas's lemma yields a slightly better conclusion than the inclusion (48): there exists a bounded operator $\mathfrak{R} : \ell_{\beta B}^2 \rightarrow L_{\beta, \Gamma_\beta B + \epsilon}^2$ such that $i = \mathcal{B}\mathfrak{R}$. In other words, the interpolation problem in the Gevrey class of order β can be solved continuously and linearly. Compare with [mitjagin'bases].

To establish the optimality of the factor Γ_β we collect some results on concentrated functionals. Our starting point is the following observation: let f be a concentrated functional on $C_{\beta B}^\infty$ for some $B > 0$. It is tempting to write

$$f = \sum_{n=0}^{\infty} f_n \delta^{(n)}, \quad (49)$$

where (f_n) are the coefficients of the entire function Φ built in Lemma A.4. However, for a sequence (a_n) of complex numbers and $\phi \in C_{\beta B}^\infty$, we are only able to make the series $\sum a_n \phi^{(n)}(0)$ convergent under the assumption

$$\sum_{n=0}^{\infty} |a_n| B^n n^{n\beta} < \infty.$$

So from (43), we are only able to make $\sum f_n \delta^{(n)}$ converge in $(C_{\beta A}^\infty)'$, for $A < B/\Gamma_\beta$. This suggests that (49) is not the optimal representation of f , and also note that we have not yet proven that this is a valid representation.

Proposition A.8. *Fix $B > 0$ and $\beta > 1$. We introduce a function h in $C_{\gamma A}^\infty$ for some $1 < \gamma < \beta$ and $A > 0$, that is equal to 1 around 0 and flat at ± 1 . We consider the weight*

$$\omega(\xi) := \exp \left(\frac{\beta}{e B^{1/\beta}} |\xi|^{1/\beta} \right),$$

and a function $\Psi \in L^2(\mathbb{R}, \omega^{-2}(\xi) d\xi)$, that is moreover entire of order $\rho < 1$ on the complex plane. Then we have

1. *The relation*

$$\langle g, \phi \rangle := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[h\phi](\xi) d\xi \quad (50)$$

defines a concentrated functional g on $L^2_{\beta B}$, which does not depend on h . Moreover, we have the estimate

$$c \|g\|_{(L^2_{\beta B})'} \leq \|\Psi\|_{L^2(\mathbb{R}, \omega^{-2}(\xi) d\xi)} \leq C \|g\|_{(L^2_{\beta B})'}, \quad (51)$$

where the constants c, C do not depend on Ψ .

2. *Let f be a concentrated functional on $L^2_{\beta B}$. The function Φ constructed in Lemma A.4 lies in $L^2(\mathbb{R}, \omega^{-2}(\xi) d\xi)$. Moreover, the functional g defined by (50) with $\Psi = \Phi$ equals f , and for every $A < B/\Gamma_\beta$ there holds*

$$f = \sum_{n=0}^{\infty} f_n(-i)^n \delta^{(n)} \quad \text{on } C^\infty_{\beta A}.$$

Remark A.9. In particular, this Proposition implies the following. Let $\Psi \in L^2(\mathbb{R}, \omega^{-2}(\xi) d\xi)$ be entire of order $\rho < 1$, then Ψ satisfies the estimate (42). This can be seen as a weighted version of the Plancherel-Pólya inequality [**plancherel'polya**].

Proof. Step 1: We show that (50) defines a functional on $L^2_{\beta B}$. For a function $\phi \in C^\infty_{\beta B}$, the function $\varphi := h\phi$ lies again in $L^2_{\beta B}$ and satisfies

$$\|\varphi\|_{L^2_{\beta B}} \lesssim \|\phi\|_{L^2_{\beta B}},$$

see [**reachable'martin**]. Moreover, since h is compactly supported, the function φ can be extended to the whole real line, satisfying the same Gevrey estimate. More precisely, we have

$$\|\varphi\|_{L^2_{\beta B}} = \|\varphi\|_{\mathcal{G}_{\beta, \hat{B}, 0}}, \quad \hat{B} := \frac{\beta}{eB^{1/\beta}}.$$

We deduce that

$$\|\varphi\|_{\mathcal{G}_{\beta, \hat{B}, 0}} = \|\varphi\|_{L^2_{\beta B}} \lesssim \|\phi\|_{L^2_{\beta B}},$$

and from Theorem 1.7 we find that

$$\|\mathcal{F}\varphi\|_{L^2(\mathbb{R}, \omega^2(\xi) d\xi)} = \|\mathcal{F}\varphi\|_{\hat{\mathcal{G}}_{\beta, \hat{B}, 0}} \lesssim \|\varphi\|_{\mathcal{G}_{\beta, \hat{B}, 0}} \lesssim \|\phi\|_{L^2_{\beta B}}.$$

From the Cauchy-Schwarz inequality, we deduce that the integral (50) thus converges, that (50) defines a functional on $L^2_{\beta B}$, and the first estimate in (51) follows.

Step 2: From now on we assume that the order ρ of Ψ satisfies $\rho < 1/\gamma$ (it will be proved in Step 4 that we have indeed $\rho \leq 1/\beta$). For the moment, we can assume that $\rho < 1/\gamma$ by choosing such γ and fixing h as in the statement of the Proposition. We show that

$$g = \sum_{n=0}^{\infty} a_n(-i)^n \delta^{(n)}, \quad \text{on } C^\infty_{\beta A} \quad (52)$$

holds for every $A < B/\Gamma_\beta$, where (a_n) are the Taylor coefficients of Ψ . To this end, we fix $A < B/\Gamma_\beta$, and observe that both functionals involved in (52) are continuous over $C^\infty_{\beta A}$ where the set C^∞_γ is dense. Hence we only have to show the equality for test functions in C^∞_γ . Let $\phi \in C^\infty_{\gamma D}$ for some $D > 0$ and compute

$$\langle g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[\phi h](\xi) d\xi$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{n=0}^{\infty} a_n \xi^n \mathcal{F}[\phi h](\xi) d\xi \\
&= \sum_{n=0}^{\infty} a_n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi^n \mathcal{F}[\phi h](\xi) d\xi \\
&= \sum_{n=0}^{\infty} a_n (-i)^n (\phi h)^{(n)}(0) \\
&= \sum_{n=0}^{\infty} a_n (-i)^n \phi^{(n)}(0),
\end{aligned}$$

where at the third equality, the interchange of the sum and the integral has to be justified. For $n \in \mathbb{N}$ we compute, with $\varphi := h\phi$,

$$\begin{aligned}
\int_{\mathbb{R}} |\xi^n \mathcal{F}\varphi(\xi)| d\xi &= \|\mathcal{F}\varphi^{(n)}\|_{L^1(\mathbb{R})} \\
&\lesssim \|\varphi^{(n)}\|_{H^1(\mathbb{R})} \\
&\lesssim \|\varphi^{(n)}\|_{C^1[-1,1]} \\
&\lesssim D^n n^{\gamma} + D^{n+1} (n+1)^{(n+1)\gamma}.
\end{aligned}$$

Because Ψ has order $\rho < 1/\gamma$, for any fixed λ with $1/\rho > \lambda > \gamma$, we have

$$\sum_{n=0}^{\infty} |a_n| n^{n\lambda} < \infty,$$

see [boas]. Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} |a_n| \int_{\mathbb{R}} |\xi^n \mathcal{F}[\phi h](\xi)| d\xi \\
&\lesssim \sum_{n=0}^{\infty} |a_n| \left[D^n n^{\gamma} + D^{n+1} (n+1)^{(n+1)\gamma} \right] \\
&\lesssim \sum_{n=0}^{\infty} |a_n| n^{n\lambda} < \infty.
\end{aligned}$$

This shows the equality (52).

Step 3: Let us show that g is a concentrated functional on $L^2_{\beta B}$. Take $\phi \in L^2_{\beta B}$ which vanishes identically on $[-\delta, \delta]$ for some $\delta > 0$. We introduce a mollifying kernel $(\varrho_\epsilon)_{\epsilon>0}$ that is Gevrey of order γ and put $\psi_\epsilon := (h\phi) * \varrho_\epsilon$. Plain computations show that $\psi_\epsilon \in C^\infty_\gamma$ together with $\psi_\epsilon \rightarrow h\phi$ in $L^2_{\beta B}$, so that from Theorem 1.7 we also have $\mathcal{F}\psi_\epsilon \rightarrow \mathcal{F}(h\phi)$ in $L^2(\mathbb{R}, \omega^2(\xi)d\xi)$. We observe that

$$\text{supp } \psi_\epsilon \subset [-1, -\delta + \epsilon] \cup [\delta - \epsilon, 1],$$

which stays away from 0 for small enough ϵ . For such ϵ , a computation identical to that made at Step 2 brings

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\psi_\epsilon(\xi) d\xi = \sum_{n=0}^{\infty} a_n (-i)^n \psi_\epsilon^{(n)}(0) = 0,$$

and passing to the limit $\epsilon \rightarrow 0$ (which is valid since $\mathcal{F}\psi_\epsilon \rightarrow \mathcal{F}(h\phi)$ in $L^2(\mathbb{R}, \omega^2(\xi)d\xi)$ and $\Psi \in L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$), we find that

$$0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[h\phi](\xi) d\xi = \langle g, \phi \rangle,$$

which shows that g is concentrated.

In particular, owing to the inclusion $C_{\beta B}^\infty \subset L_{\beta B}^2$, the functional g is a concentrated functional on $C_{\beta B}^\infty$ and we are allowed to consider the entire function Φ constructed in Lemma A.4. Let us then show that $\Phi = \Psi$. We observe that for fixed $s \in \mathbb{C}$, the function $t \mapsto e^{ist}$ is analytic on the real line, hence we may test (52) against it. This yields

$$\Phi(s) = \langle g(t), e^{ist} \rangle = \sum_{n=0}^{\infty} a_n s^n = \Psi(s).$$

Step 4: The function $\Psi = \Phi$ therefore satisfies the estimates of Lemma A.4, and in particular it has order $\rho \leq 1/\beta$. This justifies that we may fix $1 < \gamma < \beta$, and therefore h , independently of Ψ . This also shows that g does not depend on h .

Step 5: Let us show the second estimate in (51). We compute

$$\|g\|_{(L_{\beta B}^2)'} = \sup_{\|\phi\|_{L_{\beta B}^2}=1} |\langle g, \phi \rangle| = \sup_{\|\phi\|_{L_{\beta B}^2}=1} \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[h\phi](\xi) d\xi \right|,$$

and acknowledge the following fact, which has already been used in this proof: if $\psi \in \mathcal{G}_{\beta, \frac{\beta}{eB^{1/\beta}}, 0}$, then its restriction to $[-1, 1]$ lies in $L_{\beta B}^2$. To simplify the notation we use the shorthand $\mathcal{G}$ for the previous space. Let us make the following computations that we will justify later on

$$\begin{aligned} \sup_{\|\phi\|_{L_{\beta B}^2}=1} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[h\phi](\xi) d\xi \right| &\geq \sup_{\phi \in \mathcal{G}: \|\phi\|_{L_{\beta B}^2}=1} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}[h\phi](\xi) d\xi \right| \\ &= \sup_{\phi \in \mathcal{G}: \|\phi\|_{L_{\beta B}^2}=1} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\phi(\xi) d\xi \right| \\ &= \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0 \text{ on } [-1, 1]}} \frac{1}{\|\phi\|_{L_{\beta B}^2}} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\phi(\xi) d\xi \right| \\ &\geq \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0 \text{ on } [-1, 1]}} \frac{1}{\|\phi\|_{\mathcal{G}}} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\phi(\xi) d\xi \right| \\ &\geq \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0 \text{ on } \mathbb{R}}} \frac{1}{\|\phi\|_{\mathcal{G}}} \left| \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\phi(\xi) d\xi \right| \\ &\gtrsim \|\Psi\|_{L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)}. \end{aligned}$$

The first equality is due to the fact that the functional

$$\tilde{g} : \phi \mapsto \int_{\mathbb{R}} \Psi(\xi) \mathcal{F}\phi(\xi) d\xi,$$

is a concentrated functional on $\mathcal{G}$, which can be shown reasoning similarly as in Step 3. The second inequality is due to $\|\phi\|_{L_{\beta B}^2} \leq \|\phi\|_{\mathcal{G}}$. The third inequality is because $\tilde{g}$ is concentrated on $\mathcal{G}$. The last inequality is due to the Plancherel-Gevrey

Theorem 1.7, which asserts that the Fourier transform is a topological isomorphism $\mathcal{G} \rightarrow L^2(\mathbb{R}, \omega^2(\xi)d\xi)$. This shows the second estimate in (51).

Step 6: To end the proof of the Proposition, it remains to show its second item. Indeed, let f be a concentrated functional on $L^2_{\beta B}$, going back to the proof of Lemma A.4 we see that Φ satisfies the following estimate on the real line

$$\forall \delta > 0, \quad \exists C_\delta > 0, \quad \forall \xi \in \mathbb{R}, \quad |\Phi(\xi)| \leq C_\delta \exp\left(\frac{\beta}{e} \left(\frac{1+\delta}{B} |\xi|\right)^{1/\beta}\right). \quad (53)$$

Thus, Φ is nearly in the weighted Lebesgue space $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$. In fact, for fixed $0 < \tilde{B} < B$, consider the weight

$$\tilde{\omega}(\xi) := \exp\left(\frac{\beta}{e\tilde{B}^{1/\beta}} |\xi|^{1/\beta}\right).$$

From (53) we deduce that $\Phi \in L^2(\mathbb{R}, \tilde{\omega}^{-2}(\xi)d\xi)$, hence $\tilde{g}$ defined by the relation (50) with $\Psi = \Phi$ defines a concentrated functional on $L^2_{\beta\tilde{B}}$. Clearly, the functional $\tilde{g}$ does not depend on $\tilde{B}$, hence we denote it by g . For $\phi \in C^\infty_{\beta\tilde{B}}$ we compute

$$\begin{aligned} \langle g, \phi \rangle &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(\xi) \mathcal{F}[\phi h](\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \langle f(t), e^{i\xi t} \rangle \mathcal{F}[h\phi](\xi) d\xi \\ &= \left\langle f(t), \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi t} \mathcal{F}[h\phi](\xi) d\xi \right\rangle \\ &= \langle f, \phi h \rangle = \langle f, \phi \rangle. \end{aligned}$$

This shows that $g = f$ on $C^\infty_{\beta\tilde{B}}$. We deduce that g is a concentrated functional on $L^2_{\beta B}$, so that Φ lies indeed in $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$. The representation of f as a series of Diracs follows from Step 2, which ends the proof. $\square$

We are now in position to show the optimality of the factor Γ_β .

Theorem A.10. *For every $\epsilon, B > 0$ and $\beta > 1$ with $\epsilon < \Gamma_\beta B$, there exists a sequence $(a_n) \in K(n^{n\beta} B^n)$ which can be interpolated by no $\phi \in C^\infty_{\beta, \Gamma_\beta B - \epsilon}$.*

Proof. Assume by contradiction that this is not true, then fix $\epsilon, B > 0$ and $\beta > 1$ so that

$$\text{Range } \mathcal{B}^\epsilon_\infty \supset \text{Range } i^\epsilon_\infty,$$

where

$$\mathcal{B}^\epsilon_\infty : C^\infty_{\beta, \Gamma_\beta B - \epsilon} \rightarrow K(n^{n\beta}(\Gamma_\beta B - \epsilon)^n), \quad i^\epsilon_\infty : K(n^{n\beta} B^n) \rightarrow K(n^{n\beta}(\Gamma_\beta B - \epsilon)^n).$$

Similarly as in the proof of Theorem A.3, we deduce that

$$\text{Range } \mathcal{B} \supset \text{Range } i, \quad (54)$$

where

$$\mathcal{B} : L^2_{\beta, \Gamma_\beta B - \epsilon/2} \rightarrow \ell^2_{\beta, \Gamma_\beta B}, \quad i : \ell^2_{\beta, B} \rightarrow \ell^2_{\beta, \Gamma_\beta B}.$$

Denote $X = L^2_{\beta, \Gamma_\beta B - \epsilon/2}$, from (54) and Douglas's Lemma we have

$$\exists c > 0, \quad \forall a \in \ell^2_{\beta, \Gamma_\beta B}, \quad \|\mathcal{B}^* a\|_{X'} \geq c \|i^* a\|_{\ell^2_{\beta B}}. \quad (55)$$

Fix a concentrated functional f on $L_{\beta B}^2$ and consider the weight

$$\omega(\xi) := \exp\left(\frac{\beta}{e\tilde{B}^{1/\beta}}|\xi|^{1/\beta}\right), \quad \tilde{B} := \Gamma_\beta B - \epsilon/2.$$

From Proposition A.8, the entire function Φ constructed in Lemma A.4 belongs to $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$, and it has order $\leq 1/\beta$ on the complex plane. From [Borichev], there exists a sequence (P_j) of polynomials such that $P_j \rightarrow \Phi$ for the $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$ -norm. For every j , we put

$$P_j(s) =: \sum_{n=0}^{\infty} f_n^j s^n, \quad \hat{P}_j := \sum_{n=0}^{\infty} f_n^j (-i)^n \delta^{(n)}, \quad a_n^j := n^{2n\beta} (\Gamma_\beta B)^{2n} f_n^j (-i)^n,$$

so that $\hat{P}_j = \mathcal{B}^* a^j$. We apply (55) to the sequence a^j , which is finitely supported, to obtain

$$\begin{aligned} \sup_{n \in \mathbb{N}} n^{n\beta} B^n |f_n^j| &\leq \sqrt{\sum_{n=0}^{\infty} |n^{n\beta} B^n f_n^j|^2} \\ &= \|i^* a^j\|_{\ell_{\beta B}^2} \\ &\leq \frac{1}{c} \|\mathcal{B}^* a^j\|_{X'} \lesssim \|P_j\|_{L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)}. \end{aligned} \quad (56)$$

Because (P_j) is a convergent sequence of $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$, the right-hand side above is bounded with respect to j . Because P_j converges to Φ for the $L^2(\mathbb{R}, \omega^{-2}(\xi)d\xi)$ -norm we deduce that $\hat{P}_j$ converges to f for the X' -norm, and in particular we have

$$\forall n \in \mathbb{N}, \quad f_n^j \xrightarrow{j \rightarrow \infty} f_n.$$

Passing to the limit in (56) we deduce that (f_n) satisfies

$$\exists c > 0, \quad \forall n \in \mathbb{N}, \quad |f_n| \leq c B^n n^{-n\beta}, \quad (57)$$

an improvement over (43) from Lemma A.4.

Now consider an arbitrary entire function

$$\Psi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

of order < 1 on $\mathbb{C}$, satisfying

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}, \quad |\Psi(\xi)| \leq C \exp\left(\frac{\beta}{eA^{1/\beta}}|\xi|^{1/\beta}\right), \quad (58)$$

for some $A > \Gamma_\beta B - \epsilon/2$. From such Ψ we construct a concentrated functional g on $L_{\beta A}^2$ as in Proposition A.8. From the previously established fact, the coefficients (a_n) of Ψ satisfy the estimate (57). In particular, we have

$$\exists C > 0, \quad \forall z \in \mathbb{C}, \quad |\Psi(z)| \leq C \exp\left(\frac{\beta}{eB^{1/\beta}}|z|^{1/\beta}\right).$$

Put now $A := \Gamma_\beta B - \epsilon/4 > \Gamma_\beta B - \epsilon/2$, so that we have established the following fact: any function Ψ that is entire and of order < 1 on $\mathbb{C}$, that has order $\leq 1/\beta$ and type no greater than

$$\sigma := \frac{\beta}{e(\Gamma_\beta B - \epsilon/4)^{1/\beta}},$$

on the real line, is in fact of order $\leq 1/\beta$ and type no greater than

$$\sigma' := \frac{\beta}{eB^{1/\beta}},$$

on the whole complex plane. Observe that

$$\lambda := \frac{\sigma'}{\sigma} = \left(\Gamma_\beta - \frac{\epsilon}{4B} \right)^{1/\beta} < \Gamma_\beta^{1/\beta}.$$

By a scaling argument we find that: any function Ψ that is entire and of order < 1 on $\mathbb{C}$, and that has order $\leq 1/\beta$ and type $\leq K$ on the real line, is of order $\leq 1/\beta$ and type $\leq \lambda K$ on $\mathbb{C}$. To arrive to a contradiction we consider the Mittag-Leffler function

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)},$$

which is entire on $\mathbb{C}$. From Stirling's formula it is clear that E_β has order $1/\beta$ and type 1 on $\mathbb{C}$ (see [boas]). Moreover, the function E_β has the following asymptotic behaviour:

$$|E_\beta(iy)| \sim \frac{1}{\beta} \exp \left(\cos \left(\frac{\pi}{2\beta} \right) |y|^{1/\beta} \right), \quad \mathbb{R} \ni y \rightarrow \pm\infty,$$

see [mittag'leffler], hence it has type $\cos(\pi/2\beta)$ on the imaginary axis. Taking $\Psi(s) = E_\beta(is)$ and invoking the previously deduced fact we obtain: E_β has order $1/\beta$ and type $\lambda \cos(\pi/2\beta)$ on $\mathbb{C}$. Because

$$\lambda \cos \left(\frac{\pi}{2\beta} \right) < \Gamma_\beta^\beta \cos \left(\frac{\pi}{2\beta} \right) = 1,$$

we arrive to a contradiction, as E_β has type 1 on $\mathbb{C}$. $\square$

Appendix B. Proof of the Lemma on the discrete version of the Laplace method. This section is devoted to the proof of Lemma 2.1. The proof involves several computations so we divide it into several steps to make the proof easier to read.

Step 0: There is no loss of generality in assuming that $u(x_0) = 0$, and thus we aim to show that

$$\sum_{k=0}^n e^{-nu(k/n)} \sim \sqrt{\frac{2\pi n}{u''(x_0)}}, \quad n \rightarrow \infty.$$

Step 1: For every $a > 0$ and $b \in \mathbb{R}$, we claim that

$$\sum_{k=-\infty}^{+\infty} \exp \left[-n \frac{a}{2} \left(\frac{k}{n} - b \right)^2 \right] = \sqrt{\frac{2n\pi}{a}} \left[1 + O \left(e^{-\frac{2n\pi^2}{a}} \right) \right], \quad (59)$$

where the O term is uniform with respect to the parameters $n \geq 1$, $b \in \mathbb{R}$, and $a > \underline{a}$ where $\underline{a} > 0$ is any given positive number. To show this, we follow the proof of [laplace'discrete]. We write

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \exp \left[-n \frac{a}{2} \left(\frac{k}{n} - b \right)^2 \right] &= e^{-n \frac{ab^2}{2}} \sum_{k=-\infty}^{+\infty} e^{-\frac{a}{2n} k^2} e^{abk} \\ &= e^{-n \frac{ab^2}{2}} \Theta \left(\frac{ab}{2i\pi}, -\frac{a}{2in\pi} \right), \end{aligned}$$

where Θ is Jacobi's theta function, defined by

$$\Theta(z, \tau) = \sum_{k=-\infty}^{+\infty} e^{i\pi k^2 \tau} e^{2i\pi k z}, \quad z \in \mathbb{C}, \quad \text{Im } \tau > 0.$$

The functional equation for Θ gives

$$\Theta\left(z, -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} e^{i\pi \tau z^2} \Theta(z\tau, \tau),$$

see [mumford]. Applying this formula with $\tau = 2in\pi/a$ and $z = ab/2i\pi$ yields

$$e^{-n \frac{ab^2}{2}} \Theta\left(\frac{ab}{2i\pi}, -\frac{a}{2in\pi}\right) = e^{-n \frac{ab^2}{2}} \Theta\left(z, -\frac{1}{\tau}\right) = \sqrt{\frac{2n\pi}{a}} \sum_{k=-\infty}^{+\infty} e^{-\frac{2\pi^2 n}{a} k^2} e^{2i\pi k b n}.$$

In the series of the right-hand side, the term $k = 0$ is dominant:

$$\left| \sum_{k \neq 0} e^{-\frac{2\pi^2 n}{a} k^2} e^{2i\pi k b n} \right| \leq 2 \sum_{k=1}^{\infty} e^{-\frac{2\pi^2 n}{a} k^2} \leq C e^{-\frac{2\pi^2 n}{a}},$$

for some constant $C > 0$ independent of $n \geq 1$ and $a > \underline{a}$. This shows the claim.

Step 2: We fix $\delta > 0$, $n \geq 1$ and we put

$$m(\delta) = \inf_{|x-x_0| \geq \delta} u(x),$$

which is positive (u has the unique global minimum 0). We deduce that

$$\sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| \geq \delta}}^n e^{-nu(k/n)} = O\left(ne^{-nm(\delta)}\right). \quad (60)$$

We then deduce, thanks to (60), that

$$\begin{aligned} \sum_{k=0}^n e^{-nu(k/n)} &= \sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| < \delta}}^n e^{-nu(k/n)} + \sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| \geq \delta}}^n e^{-nu(k/n)} \\ &= \sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| < \delta}}^n e^{-nu(k/n)} + O\left(ne^{-nm(\delta)}\right), \end{aligned} \quad (61)$$

and estimate the sum appearing on the right-hand side. We estimate it from above, and we will estimate it from below in the next step. We put

$$\ell(\delta) = \inf_{|x-x_0| < \delta} u''(x),$$

which is well defined and positive for small enough δ , which will be assumed henceforth, and which converges to $u''(x_0)$ as $\delta \rightarrow 0^+$. This allows

$$|x - x_0| < \delta \implies u(x) \geq \frac{\ell(\delta)}{2} (x - x_0)^2,$$

hence, by (59),

$$\sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| < \delta}}^n e^{-nu(k/n)} \leq \sum_{\substack{k=0 \\ |\frac{k}{n}-x_0| < \delta}}^n e^{-n \frac{\ell(\delta)}{2} (\frac{k}{n}-x_0)^2}$$

$$\begin{aligned}
&\leq \sum_{k=-\infty}^{+\infty} e^{-n \frac{\ell(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} \\
&= \sqrt{\frac{2n\pi}{\ell(\delta)}} \left[1 + O\left(e^{-\frac{2n\pi^2}{\ell(\delta)}}\right) \right], \tag{62}
\end{aligned}$$

where the O term is uniform in $n \geq 1$ and $\delta > 0$ small enough. Thus, for $\delta > 0$ fixed and small enough, we have

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{u''(x_0)}{2n\pi}} \sum_{k=0}^n e^{-nu(k/n)} \leq \sqrt{\frac{u''(x_0)}{\ell(\delta)}}.$$

Letting $\delta \rightarrow 0^+$ in the right-hand side, we find that

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{u''(x_0)}{2n\pi}} \sum_{k=0}^n e^{-nu(k/n)} \leq 1. \tag{63}$$

Step 3: The estimate from below is slightly more tedious. We put, for $\delta > 0$ small enough so that $[x_0 - \delta, x_0 + \delta] \subset (0, 1)$,

$$L(\delta) = \sup_{|x-x_0| < \delta} u''(x) > 0,$$

so that by (59),

$$\begin{aligned}
\sum_{\substack{k=0 \\ |\frac{k}{n} - x_0| < \delta}}^n e^{-nu(k/n)} &\geq \sum_{\substack{k=0 \\ |\frac{k}{n} - x_0| < \delta}}^n e^{-n \frac{L(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} \\
&= \left\{ \sum_{k=-\infty}^{+\infty} - \sum_{k=n+1}^{\infty} - \sum_{k=-\infty}^{-1} - \sum_{\substack{k=0 \\ |\frac{k}{n} - x_0| \geq \delta}}^n \right\} e^{-n \frac{L(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} \\
&= \sqrt{\frac{2n\pi}{L(\delta)}} \left[1 + O\left(e^{-\frac{2n\pi^2}{L(\delta)}}\right) \right] - \\
&\quad - \left\{ \sum_{k=n+1}^{\infty} + \sum_{k=-\infty}^{-1} \right\} e^{-n \frac{L(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} + \\
&\quad + O\left(n e^{-n \frac{L(\delta)}{2} \delta^2}\right). \tag{64}
\end{aligned}$$

We are thus left to estimate the remaining two series in the right-hand side of (64). Until the end of the proof, κ is any positive number which depends on δ and x_0 , changing from line to line, small enough so that the written asymptotics are valid. To bound above the first series we use a comparison between series and integrals:

$$\sum_{k=n+1}^{\infty} e^{-n \frac{L(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} \leq \int_n^{\infty} e^{-\kappa n \left(\frac{x}{n} - x_0\right)^2} dx = \sqrt{\frac{n}{\kappa}} \int_{(1-x_0)\sqrt{n\kappa}}^{\infty} e^{-x^2} dx = O(e^{-\kappa n}).$$

For the other series we compute

$$\sum_{k=-\infty}^{-1} e^{-n \frac{L(\delta)}{2} \left(\frac{k}{n} - x_0\right)^2} \leq \sum_{k=1}^{\infty} \exp \left[-\kappa \left(\frac{k^2}{n} + 2kx_0 + x_0^2 n \right) \right]$$

$$\leq e^{-\kappa n} \sum_{k=1}^{\infty} e^{-\kappa k} = O(e^{-\kappa n}).$$

Coming back to (64) we find

$$\sum_{\substack{k=0 \\ \left| \frac{k}{n} - x_0 \right| < \delta}}^n e^{-nu(k/n)} \geq \sqrt{\frac{2n\pi}{L(\delta)}} [1 + O(e^{-\kappa n})] + O(e^{-\kappa n}) \geq \sqrt{\frac{2n\pi}{L(\delta)}} + O(e^{-\kappa n}),$$

and we conclude, similarly as in the previous step, that

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{u''(x_0)}{2n\pi}} \sum_{k=0}^n e^{-nu(k/n)} \geq 1,$$

which, together with (63), concludes the proof of the Lemma.

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