Hybrid range inclusions: a generalization of Douglas' theorem, with an application to control theory

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Abstract

Douglas' majorization and factorization theorem characterizes the inclusion of operator ranges in Hilbert spaces. It notably reinforces the well-established connections between the inclusion of kernels of operators in Hilbert spaces and the (inverse) inclusion of the closures of the ranges of their adjoints. This article aims to present a new "mixed" version of these concepts, for operators whose codomain lies in a product space. Additionally, we present an application to the control theory of coupled linear partial differential equations.

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1 Introduction

Let us set some useful standard notations. $\mathbb{N}^*$ denotes $\mathbb{N}\setminus\{0\}$. For a subset E in a metric space, $\overline{E}$ is the closure of E. For an operator T between linear spaces, $\mathcal{R}(T)$ is the range of T, and $\mathcal{N}(T)$ is the kernel of T. For E and F two normed vector spaces, $\mathcal{L}_c(E,F)$ is the set of linear continuous maps from E to F (endowed with the operator norm $|||\cdot|||$), whereas $\mathcal{L}(E,F)$ is the set of linear (and not necessarily continuous) maps from E to F.

1.1 Douglas' majorization and factorization theorem and a weaker analogue

Let H_1, H_2, H_3 be (all) real or (all) complex Hilbert spaces, endowed with some scalar or hermitian product (assumed to be left-linear in the complex case), with associated norms $||\cdot||_i$ ($i=1,2,3$). Consider two operators $A \in \mathcal{L}_c(H_1, H_3)$ and $B \in \mathcal{L}_c(H_2, H_3)$. Then, we have the following so-called Douglas' majorization and factorization theorem.

Theorem 1.1 (Douglas' majorization and factorization theorem, [5])**.** *The following statements are equivalent:*

(i) (range inclusion) $\mathcal{R}(A) \subset \mathcal{R}(B)$.

(ii) (majorization) There exists $c > 0$ *such that for any* $z \in H_3$*, we have*

$$||A^*z||_1 \leqslant c||B^*z||_2.$$

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(iii) (factorization) $A = BC$ for some $C \in \mathcal{L}_c(H_1, H_2)$, or, equivalently, $A^* = DB^*$ for some $D \in \mathcal{L}_c(H_2, H_1)$.

The principal interest of Theorem 1.1 is that it gives some equivalent (and somehow more convenient) reformulations of the inclusion of operator ranges in Hilbert spaces and the factorization of operators. In particular, the initial problem (i) in Theorem 1.1 is an *existence problem*, and Point (ii) reformulates it in terms of a *uniqueness problem*, under the form of an appropriate inequality to be proved, which turns out to be often mathematically much more convenient.

Since its discovery, this theorem has proved to be a powerful tool in general operator theory, has been extensively used, and even partially generalized to reflexive Banach spaces [6], and in general topological vector spaces [2] or Hilbert C^*-modules [12], at least under some additional assumptions. See also [5] or [10] for generalizations on certain unbounded operators. Let us also mention that Douglas' majorization and factorization theorem also turns out to have many other applications. We can mention interpolation in vector-valued Reproducing Kernel Hilbert Spaces (see [11, Section 6.3]), properties of Hankel or Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ (see for instance the recent paper [3] and the references therein), or abstract linear control systems in infinite dimensional spaces (see the seminal paper [4]), among many others.

Alternatively, Douglas' majorization and factorization theorem can also be reinterpreted as a reinforcement of well-known properties related to the inclusions of the closures of operator ranges and their link with orthogonality, under the form of a "non-quantitative majorization" and "non-continuous factorization" theorem, which we will refer to as the "weak majorization and factorization theorem", whose proof is standard.

Theorem 1.2 (weak majorization and factorization theorem)**.** *The following statements are equivalent:*

(i) (range-closure inclusion) $\overline{\mathcal{R}(A)} \subset \overline{\mathcal{R}(B)}$ (or, equivalently, $\mathcal{R}(A) \subset \overline{\mathcal{R}(B)}$, or equivalently, for any $h_1 \in H_1$ and any $\varepsilon > 0$, there exists $h_2 \in H_2$ such that $||Ah_1 - Bh_2||_3 \leqslant \varepsilon$).

(ii) (non-quantitative majorization) $\mathcal{N}(B^*) \subset \mathcal{N}(A^*)$, i.e.

$$\forall z \in H_3,\ B^*z = 0 \Rightarrow A^*z = 0.$$

(iii) (non-continuous factorization) $A^* = DB^*$ for some $D \in \mathcal{L}(H_2, H_1)$ not necessarily continuous.

Remark 1.3. *Since D need not be continuous here, the factorization property $A = BC$ for some $C \in \mathcal{L}(H_1, H_2)$ not necessarily continuous cannot be derived by passing to the adjoint. This is reasonable, since $A = BC$ implies that $\mathcal{R}(A) \subset \mathcal{R}(B)$, even if C is not continuous, and this property is clearly stronger than the desired one $\mathcal{R}(A) \subset \overline{\mathcal{R}(B)}$ in infinite dimensions (consider for instance a non-onto operator B with dense range). In particular, Theorem 1.1 shows that if $A = BC_1$ for some possibly non-continuous C_1, then $A = BC_2$ for some continuous C_2.*

We do not claim originality for both of Theorems 1.1 and 1.2. We recall them to underline that their formulations are quite similar, and because they are the building blocks for proving our main and more elaborate theorem (Theorem 2.1), that is new, to the best of our knowledge.

We will examine in Section 1.2 operators with codomain in a product Hilbert space, and we will notably present the main question of interest in this article (Goal 1.4), that has not been investigated, to the best of our knowledge. This will enable us to derive a new "mixed" version of Douglas' and weak majorization and factorization theorems (our main Theorem 2.1). The necessary and sufficient conditions given in this theorem are not really easy to manipulate, so we give some sufficient condition (Proposition 2.4) that might be easier to verify in practice.

As already mentioned, Douglas' majorization and factorization theorem is an essential tool for studying abstract linear control systems in infinite dimensional spaces. The applications we have in mind come from control theory for coupled systems of linear partial differential equations.(see *e.g.* [8, Theorem 1.10], where the kind of mixed version presented later on appears naturally), but our hope is that the present article might also be helpful in other fields of mathematics. In particular, we present in Section 3 an application to a toy model of coupled heat system. Further advanced applications in control theory will be discussed elsewhere.

1.2 A mixed version for a range in a product space

Now assume $H_3 = H_4 \times H_5$, where H_4, H_5 are some Hilbert spaces on the same field as H_1 and H_2, endowed with some scalar or hermitian product, with associated norms $||\cdot||_i$ $(i = 4, 5)$. As usual, we endow in this case H_3 with the Hilbertian norm

$$||(h_4, h_5)||^2_{H_3} = ||h_4||^2_{H_4} + ||h_5||^2_{H_5}, \ (h_4, h_5) \in H_4 \times H_5.$$

Then, for $h_i \in H_i$ $(i = 1, 2, 4, 5)$, one can write

$$A(h_1) = \begin{pmatrix} A_1(h_1) \\ A_2(h_1) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} h_1,$$

with $A_1 \in \mathcal{L}_c(H_1, H_4)$, $A_2 \in \mathcal{L}_c(H_1, H_5)$, or equivalently

$$A^*(h_4, h_5) = A_1^*(h_4) + A_2^*(h_5) = \begin{pmatrix} A_1^* & A_2^* \end{pmatrix} \begin{pmatrix} h_4 \\ h_5 \end{pmatrix},$$

and similarly for B:

$$B(h_2) = \begin{pmatrix} B_1(h_2) \\ B_2(h_2) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} h_2,$$

with $B_1 \in \mathcal{L}_c(H_2, H_4)$, $B_2 \in \mathcal{L}_c(H_2, H_5)$, or equivalently

$$B^*(h_4, h_5) = B_1^*(h_4) + B_2^*(h_5) = \begin{pmatrix} B_1^* & B_2^* \end{pmatrix} \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}.$$

Our main result will be a kind of "mixed" usual-weak Douglas' majorization and factorization theorem: for the first component of A, we ask for an exact range inclusion (as in Theorem 1.1), but for the second component, we ask for some approximate inclusion of range (as in Theorem 1.2). More precisely, our objective is to characterize the following property.

Goal 1.4. *For any $h_1 \in H_1$ and any $\varepsilon > 0$, find $h_2 \in H_2$ such that $A_1 h_1 = B_1 h_2$ and $||A_2 h_1 - B_2 h_2||_5 \leqslant \varepsilon$.*

The key point is that a single h_2 must achieve both objectives. Clearly, this has to be more restrictive than the following properties considered separately:

- For any $h_1 \in H_1$, find $h_2 \in H_2$ such that $A_1 h_1 = B_1 h_2$.
- For any $h_1 \in H_1$ and any $\varepsilon > 0$, find $\tilde{h}_2 \in H_2$ such that $||A_1 h_1 - B_1 \tilde{h}_2||_4 \leqslant \varepsilon$ and $||A_2 h_1 - B_2 \tilde{h}_2||_5 \leqslant \varepsilon$, which notably implies that $||A_1 h_1 - B_1 \tilde{h}_2||^2_4 + ||A_2 h_1 - B_2 \tilde{h}_2||^2_5 \leqslant 2\varepsilon^2$, *i.e.* $||Ah_1 - B\tilde{h}_2||^2_3 \leqslant 2\varepsilon^2$.

Hence, applying Theorem 1.1 to the operators (A_1, B_1) and Theorem 1.2 to the operators (A, B) yields the following natural necessary condition

Proposition 1.5. *The following two equivalent statements are necessary conditions for Goal 1.4 to hold.*
(i) There exists $c > 0$ such that for any $h_4 \in H_4$, we have

$$||A_1^* h_4||_1 \leqslant c||B_1^* h_4||_2,$$

and $\mathcal{N}(B^*) \subset \mathcal{N}(A^*)$, i.e. $\forall (h_4, h_5) \in H_3$, $B_1^* h_4 + B_2^* h_5 = 0 \Rightarrow A_1^* h_4 + A_2^* h_5 = 0$.
(ii) $A_1 = B_1 C_1$ for some $C_1 \in \mathcal{L}_c(H_1, H_2)$ (or, equivalently, $A_1^* = D_1 B_1^*$ for some $D_1 \in \mathcal{L}_c(H_2, H_1)$), and $A^* = DB^*$ for some $D \in \mathcal{L}(H_2, H_1)$ not necessarily continuous.

Remark 1.6. *In general, properties (i) and (ii) of Proposition 1.5 are not sufficient for Goal 1.4 to hold. We will provide a counterexample in Proposition 2.3. One of the goals of this article will be to understand how to fill the gap in order to find reinforcements of these necessary conditions that turn out to be also sufficient (see Theorem 2.1).*

2 Main result

We can now state our mixed version of a majorization and factorization theorem.

Theorem 2.1. *The following statements are equivalent:*

(i) (mixed range inclusion and range-closure inclusion) Goal 1.4 is achieved: for any $h_1 \in H_1$ and any $\varepsilon > 0$, there exists $h_2 \in H_2$ such that $A_1 h_1 = B_1 h_2$ and $||A_2 h_1 - B_2 h_2||_5 \leqslant \varepsilon$.

(ii) (mixed quantitative and non-quantitative majorization) The following two conditions hold.

- *There exists $c > 0$ such that for any $h_4 \in H_4$, we have*

$$||A_1^* h_4||_1 \leqslant c||B_1^* h_4||_2.$$

- *Consider any $h_5 \in H_5$ such that there exists $(y_n)_{n\in\mathbb{N}^*} \in H_4^{\mathbb{N}^*}$ for which $B_2^* h_5 = \lim_{n\to+\infty} B_1^* y_n$. Then $A_2^* h_5 = \lim_{n\to+\infty} A_1^* y_n$.*

(iii) (mixed continuous and non-continuous factorization) Denote by $\Pi : H_2 \to \mathcal{N}(B_1)$ the orthogonal projection onto $\mathcal{N}(B_1)$ in H_2. Then $A_1^* = D_1 B_1^*$ and $A_2^* = (D_1 + D_2 \Pi) B_2^*$ for some $D_1 \in \mathcal{L}_c(H_2, H_1)$ and $D_2 \in \mathcal{L}(\mathcal{N}(B_1), H_1)$ not necessarily continuous.

Remark 2.2. *(ii) and (iii) enable us to understand the gap between the necessary conditions given in Proposition 1.5 and the necessary and sufficient conditions. Note that condition (ii) in Theorem 2.1 is stronger than condition (i) in Proposition 1.5 (consider a constant sequence), and that condition (iii) in Theorem 2.1 is stronger than condition (ii) in Proposition 1.5 (by extending D_2 to 0 on $\mathcal{N}(B_1)^\perp$ so that it is now defined on H_2).*

Proof. We prove that (i) implies (iii). First, note that (i) notably implies that $\mathcal{R}(A_1) \subset \mathcal{R}(B_1)$, so we can apply Theorem 1.1 and deduce that $A_1 = B_1 C_1$ for some $C_1 \in \mathcal{L}_c(H_1, H_2)$. As already noted, (i) can be reformulated as follows: for any $\varepsilon > 0$ and any $h_1 \in H_1$, there exists $h_2 \in H_2$ such that $A_1 h_1 = B_1 h_2$ and $||A_2 h_1 - B_2 h_2||_5 \leqslant \varepsilon$. For such h_1, h_2, we have $B_1(C_1 h_1 - h_2) = 0$, so $h_2 = C_1 h_1 + u$ for some $u \in \mathcal{N}(B_1)$. Now, for fixed h_1, (i) further implies that we must also have $A_2 h_1 = B_2 h_2 = B_2 C_1 h_1 + B_2 u$, *i.e.* $||A_2 h_1 - B_2 C_1 h_1 - B_2 u||_5 \leqslant \varepsilon$ for some $u \in \mathcal{N}(B_1)$. We deduce that

$$\overline{\mathcal{R}(A_2 - B_2 C_1)} \subset \overline{\mathcal{R}(\tilde{B}_2)},$$

where $\tilde{B}_2 = (B_2)_{|\mathcal{N}(B_1)}$. Applying Theorem 1.2 implies that there exists some $D_2 \in \mathcal{L}(\mathcal{N}(B_1), H_1)$ not necessarily continuous such that

$$A_2^* - C_1^* B_2^* = D_2 \tilde{B}_2^*.$$

Noting that $\tilde{B}_2^* = \Pi B_2^*$ and setting $D_1 = C_1^*$ gives that (iii) is verified.

Now, let us prove that (iii) implies (ii). This simply amounts to observing that for any $(h_4, h_5) \in H_4 \times H_5$, we have

$$\begin{aligned} A_1^* h_4 + A_2^* h_5 &= D_1 B_1^* h_4 + (D_1 + D_2 \Pi) B_2^* h_5 \\ &= (D_1 + D_2 \Pi)(B_1^* h_4 + B_2^* h_5), \end{aligned}$$

since $B_1 \Pi = 0$ by definition of Π (so $\Pi^* B_1^* = \Pi B_1^* = 0$). Choosing $h_5 = 0$ leads to

$$||A_1^* h_4||_1 \leqslant c||B_1^* h_4||_2,$$

with $C = |||D_1|||$, which gives the first property of (ii). For the second property, it is slightly more intricate. Assume that for some $h_5 \in H_5$, we have some $(y_n)_{n\in\mathbb{N}^*} \in H_4^{\mathbb{N}^*}$ such that

$$B_2^* h_5 = \lim_{n\to+\infty} B_1^* y_n.$$

This assertion is equivalent to the fact that $B_2^* h_5 \in \overline{\mathcal{R}(B_1^*)} = \mathcal{N}(B_1)^\perp$, *i.e.*, is equivalent to the fact that $\Pi B_2^* h_5 = 0$. So, for such a h_5, we have by (iii) that $A_2^* h_5 = D_1 B_2^* h_5$, and we also have that for any $n \in \mathbb{N}^*$, we have $D_1 B_1^* y_n = A_1 y_n$. By continuity of D_1, this quantity converges as $n \to +\infty$ to $D_1 B_2^* h_5 = A_2^* h_5$, whence the desired result.

It remains to prove that (ii) implies (i). First of all, applying Theorem 1.1 to A_1 and B_1, we know that $A_1 = B_1 C_1$ for some $C_1 \in \mathcal{L}_c(H_1, H_2)$. Let us prove that

$$\mathcal{R}(A_2 - B_2 C_1) \subset \overline{\mathcal{R}(\tilde{B}_2)},$$

where $\tilde{B}_2 = (B_2)_{|\mathcal{N}(B_1)}$, *i.e.*, by passing to the orthogonal,

$$\mathcal{N}(\Pi B_2^*) \subset \mathcal{N}(A_2^* - C_1^* B_2^*).$$

Let us consider some $h_5 \in \mathcal{N}(\Pi B_2^*)$. This means that $\Pi B_2^* h_5 = 0$, *i.e.* $B_2^* h_5 \in \mathcal{N}(B_1)^\perp$, *i.e.* $B_2^* h_5 \in \overline{\mathcal{R}(B_1^*)}$. Hence, we have some $(y_n)_{n\in\mathbb{N}^*} \in H_4^{\mathbb{N}^*}$ such that

$$B_2^* h_5 = \lim_{n\to+\infty} B_1^* y_n,$$

and by hypothesis, we also have

$$A_2^* h_5 = \lim_{n\to+\infty} A_1^* y_n = \lim_{n\to+\infty} C_1^* B_1^* y_n = C_1^* B_2^* h_5.$$

Thus, we have $h_5 \in \mathcal{N}(A_2^* - C_1^* B_2^*)$, as required. We deduce that

$$\mathcal{R}(A_2 - B_2 C_1) \subset \overline{\mathcal{R}(\tilde{B}_2)}.$$

Hence, for any $\varepsilon > 0$ and any $h_1 \in H_1$, we have $||A_2 h_1 - B_2 C_1 h_1 - B_2 u||_5 \leqslant \varepsilon$ for some $u \in \mathcal{N}(B_1)$. Setting $h_2 = C_1 h_1 + u$, we have (i) as required, since $B_1 h_2 = B_1(C_1 h_1 + u) = A_1 h_1$. $\square$

Now that we have proved Theorem 2.1, we can go back to the counterexample evoked in Remark 1.6.

Proposition 2.3. *Condition (i) or (ii) in Proposition 1.5 is, in general, not sufficient for Goal 1.4 to hold.*

Proof. We provide a counterexample. Consider H any separable infinite-dimensional Hilbert space, endowed with a norm $||\cdot||$, and consider $\{e_n\}_{n\in\mathbb{N}^*}$ a Hilbert basis of H. We choose

$$H_1 = H_2 = H_4 = H_5 = H.$$

Assume for the moment that there exist selfadjoint operators B_1 and B_2 (so we can drop the adjoint on these operators in what follows) such that the closure of $(B_2)^{-1}(\mathcal{R}(B_1))$, denoted by F, is not equal to H, and such that $(B_2)^{-1}(\overline{\mathcal{R}(B_1)}) = H$. We take $A_1 = 0$, so that $A_1 = B_1C_1$ with $C_1 = 0 \in \mathcal{L}_c(H)$. We consider A_2 the orthogonal projection on $F^\perp$. Assume that $B_1h_4 + B_2h_5 = 0$ for some $(h_4, h_5) \in H^2$. We deduce that $B_2h_5 \in \mathcal{R}(B_1)$, so $h_5 \in F$. By our choice of A_2, we deduce that $A_2h_5 = 0$, so the property holds that $A_1h_4 + A_2h_5 = 0$, as required. However, we do not have the stronger property that for any $h_1 \in H_1$ and any $\varepsilon > 0$, there exists $h_2 \in H$ such that $A_1h_1 = B_1h_2$ and $||A_2h_1 - B_2h_2||_5 \leqslant \varepsilon$. Indeed, by contradiction, this would mean that (iii) of Theorem 2.1 holds. Note that any $h_5 \in H$ is such that there exists some $(y_n)_{n\in\mathbb{N}^*} \in H_4^{\mathbb{N}^*}$ for which $B_2h_5 = \lim_{n\to+\infty} B_1y_n$, since $(B_2)^{-1}(\overline{\mathcal{R}(B_1)}) = H$. We need to conclude that $A_2h_5 = 0$, *i.e.* $h_5 \in \mathcal{N}(A_2) = F$, which is not possible if we have chosen $h_5 \in F^\perp \setminus \{0\}$.

It remains to prove the existence of such B_1, B_2. Consider B_1 as the unique linear continuous operator such that $B_1(e_n) = e_n n^{-2}$ for all $n \in \mathbb{N}^*$, and $B_2(h) = \langle h, x\rangle x$, where $x = \sum_{n=1}^{+\infty} e_n n^{-1}$. B_1 is a diagonal operator on a Hilbert basis, so it is selfadjoint. B_2 is also clearly selfadjoint by using the definition of the adjoint operator. Moreover, $\mathcal{R}(B_1)$ is dense in H, so that we have $(B_2)^{-1}(\overline{\mathcal{R}(B_1)}) = H$. Then, $h \in (B_2)^{-1}(\mathcal{R}(B_1))$ is equivalent to the fact that $B_2(h) \in \mathcal{R}(B_1)$, which is equivalent to the existence of a sequence $(f_n)_{n\in\mathbb{N}^*}$ in $l^2(\mathbb{N}^*)$ such that for any $n \in \mathbb{N}^*$, we have

$$\frac{\langle h, x\rangle}{n} = \frac{f_n}{n^2}, \ \textit{i.e.}\ \langle h, x\rangle = \frac{f_n}{n}.$$

in particular, that f_n/n has necessarily to be constant, which means that $f_n = 0$, since $(f_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. We deduce that for any $h \in (B_2)^{-1}(\mathcal{R}(B_1))$, we have $\langle x, h\rangle = 0$, which implies $(B_2)^{-1}(\mathcal{R}(B_1)) \subset x^\perp$, and concludes our proof since $x^\perp$ is closed. □

Our previous theorem might be difficult to prove in practice. So, our next goal is to give an effective sufficient condition, that can be verified in practice, as we will see below. Assume that $\mathcal{N}(B^*) = 0$. Then, introduce the following Hilbertian norm on H_4:

$$||h_4||_*^2 = ||B_1^*h_4||_2^2.$$

It is a norm. Indeed, if $B_1^*h_4 = 0$, in particular, we have $B^*(h_4, 0) = 0$, and so $(h_4, 0) = 0$. We introduce the completion $\overline{H_4}^{||\cdot||_*}$ for this norm, and

$$X = \overline{H_4}^{||\cdot||_*} \times H_5,$$

endowed with the Hilbertian norm

$$||(h_4, h_5)||_{**}^2 = ||B_1^*h_4||_2^2 + ||h_5||_5^2,$$

which makes X a Hilbert space.

Then, by definition of $||\cdot||_{**}$, one can extend B^* to X as a linear continuous map, still denoted by B^*. Under these assumptions, we have the following proposition.

Proposition 2.4. *If $\mathcal{N}(B^*) = 0$, if the extension B^* on X also satisfies $\mathcal{N}(B^*) = 0$, and if there exists $c > 0$ such that for any $h_4 \in H_4$, we have*

$$||A_1^* h_4||_1 \leqslant c||B_1^* h_4||_2,$$

then, for any $h_1 \in H_1$ and any $\varepsilon > 0$, there exists $h_2 \in H_2$ such that $A_1 h_1 = B_1 h_2$ and $||A_2 h_1 - B_2 h_2||_5 \leqslant \varepsilon$.

Remark 2.5. *It might appear surprising that the condition of Proposition 2.4 does not depend on A_2. In fact, this comes from the strong assumption that $\mathcal{N}(B^*) = 0$, which implies by orthogonality that B has dense range. So, $\mathcal{R}(B)$ is so "big" that in this context, the exact form of A_2 is not important. This can also be observed in the proof.*

Proof. According to Theorem 2.1, it suffices to prove that for any $h_5 \in H_5$ such that there exists some $(y_n)_{n\in\mathbb{N}^*} \in H_4^{\mathbb{N}^*}$ for which $B_2^* h_5 = \lim_{n\to+\infty} B_1^* y_n$, then, $A_2^* h_5 = \lim_{n\to+\infty} A_1^* y_n$. Consider such a h_5. Then observe that the sequence $(y_n, 0)_{n\in\mathbb{N}^*}$ is a Cauchy sequence with respect to the $||\cdot||_{**}$-norm, so it converges to some $(z, 0) \in X$. By uniqueness, we necessarily have that $B_1^* z = B_2^* h_5$. So, we have that $(z, -h_5) \in \mathcal{N}(B^*)$, and hence $z = h_5 = 0$, which implies that $A_2^* h_5 = 0$. Since $||A_1^* y_n||_1 \leqslant c||B_1^* y_n||_2$ for any $n \in \mathbb{N}^*$, we also have $A_1^* y_n \to 0$ as $n \to +\infty$, which concludes the proof. □

3 Application: a coupled heat system on a bounded domain

3.1 Semigroup framework

Let Ω be a bounded, connected, and open subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$, of class $\mathscr{C}^2$. We first consider the following (uncontrolled) system of heat equations

$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y &=& 0 & \text{in } (0,T)\times\Omega, \\ y &=& 0 & \text{on } (0,T)\times\partial\Omega, \\ y\,(0,\cdot) &=& y^0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t} - \Delta z &=& y & \text{in } (0,T)\times\Omega, \\ z &=& 0 & \text{on } (0,T)\times\partial\Omega, \\ z\,(0,\cdot) &=& z^0 & \text{in } \Omega, \end{array}\right. \tag{3.1}$$

for some $y^0, z^0 \in L^2(\Omega)$. Let

$$H := L^2(\Omega), \qquad D(\Delta) := H^2(\Omega) \cap H_0^1(\Omega).$$

It is well known that Δ with domain $D(\Delta)$ is self-adjoint, negative, and maximal dissipative on H. In particular, Δ generates a strongly continuous semigroup $(e^{t\Delta})_{t\geqslant 0}$ on H (see, *e.g.*, [7, Proposition, p.91]).

Now, we introduce the product Hilbert space

$$\mathcal{H} := H \times H$$

endowed with its canonical norm, and we define the operator

$$\mathcal{A} := \begin{pmatrix} \Delta & 0 \\ I & \Delta \end{pmatrix}, \qquad D(\mathcal{A}) := D(\Delta) \times D(\Delta).$$

With the notation $U(t) := (y(t), z(t))^t$, system (3.1) can be rewritten as an abstract Cauchy problem on $\mathcal{H}$:

$$\begin{cases} \dot{U}(t) = \mathcal{A}U(t), \\ U(0) = (y^0, z^0)^t. \end{cases} \tag{3.2}$$

Using a standard bounded perturbation argument of semigroups (see, e.g., [7, Theorem 1.3, Chapter III], the operator $\mathcal{A}$ generates a strongly continuous semigroup $(\mathbb{T}_t)_{t\geqslant 0}$ on $\mathcal{H}$, which is in fact explicitly given by

$$\mathbb{T}_t = \begin{pmatrix} e^{t\Delta} & 0 \\ \displaystyle\int_0^t e^{(t-s)\Delta}\, ds & e^{t\Delta} \end{pmatrix}, \qquad t \geqslant 0.$$

3.2 The control problem under study

Let ω be a nonempty open subset of Ω. Let $T > 0$ bea time horizon, that can be chosen arbitrarily small. Consider the following controlled version of (3.1) given by

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y &= & \mathbb{1}_\omega h & \text{in } (0,T)\times\Omega, \\ y &= & 0 & \text{on } (0,T)\times\partial\Omega, \\ y\,(0,\cdot) &= & y^0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t} - \Delta z &= & y & \text{in } (0,T)\times\Omega, \\ z &= & 0 & \text{on } (0,T)\times\partial\Omega, \\ z\,(0,\cdot) &= & z^0 & \text{in } \Omega, \end{cases} \tag{3.3}$$

for some $y^0, z^0 \in L^2(\Omega)$ and some $h \in L^2((0,T)\times\Omega)$.

It is well known that this system has the following two properties:

- It is *null controllable* in arbitrary small time: for any $y^0, z^0 \in L^2(\Omega)$, there exists $h \in L^2((0,T)\times\Omega)$ such that $y(T) = z(T) = 0$.
- It is also *approximately controllable* in arbitrary small time: for any $y^0, z^0, y^T, z^T \in L^2(\Omega)$, for any $\varepsilon > 0$ there exists $h \in L^2((0,T)\times\Omega)$ such that $||y(T) - y^T||_{L^2(\Omega)} \leqslant \varepsilon$ and $||z(T) - z^T||_{L^2(\Omega)} \leqslant \varepsilon$.

Both results are well-known and consequences, for instance, of [1, Theorem 3].

For our purposes, we need to introduce more advanced tools in control theory. Let us call R the reachable space at time T (starting from a null initial condition), which is simply the range of the linear continuous operator

$$\Phi_T h : \left(x \mapsto \int_0^T \mathbb{T}_{T-\sigma}(\mathbb{1}_\omega(x)h(\sigma,x))\, d\sigma \right), \tag{3.4}$$

from $L^2((0,T)\times\Omega)$ into $L^2(\Omega)^2$. Then, since (3.3) is approximately controllable in time $T > 0$, according to [9, Page 6], we can endow R with a Hilbertian norm

$$||p||_R = \inf_{h\in L^2((0,T)\times\Omega),\Phi_T h = p} ||h||_{L^2((0,T)\times\Omega)},$$

for which R is a Hilbert space. Then, according to [9, Definition 5.2 and Proposition 5.8], we have

$$R \subset \mathcal{H} \subset R',$$

with continuous and dense inclusions, and

$$||q||_{R'} = ||\Phi_T^* q||_{L^2(\Omega)^2}, \forall q \in L^2(\Omega)^2.$$

Since (3.3) is null controllable in arbitrary small time, applying [9, Theorem 2.1], $(\mathbb{T}_t)_{t\geqslant 0}$ can be restricted to the reachable space R, on which it is still a strongly continuous semigroup (on the Hilbert space R), that we denote by $(\tilde{\mathbb{T}}_t)_{t\geqslant 0}$. Since R is a reflexive space, we also have an adjoint strongly continuous semigroup $(\tilde{\mathbb{T}}_t^*)_{t\geqslant 0}$ on R', that coincides with $(\mathbb{T}_t^*)_{t\geqslant 0}$ on $\mathcal{H}$.

3.3 Main result and its proof

As explained in Remark 3.2, for the question we would like to investigate in what follows, there is no loss of generality in assuming that $z^0 = 0$, so we will instead look at

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y & = & \mathbb{1}_\omega h & \text{in } (0,T)\times\Omega, \\ y & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) & = & y^0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t} - \Delta z & = & y & \text{in } (0,T)\times\Omega, \\ z & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,\cdot) & = & 0 & \text{in } \Omega, \end{array} \right. \tag{3.5}$$

which enjoys similar approximate and null controllability properties.

We now investigate a problem combining null-controllability and approximate controllability. As far as we know, this result is new.

Theorem 3.1. *For any $(y^0, z^T) \in L^2(\Omega)^2$ and any $\varepsilon > 0$, there exists $h \in H_2$ such that the solution (y,z) of (3.5) satisfies $y(T) = 0$ and $||z(T) - z^T||_{L^2(\Omega)} \leqslant \varepsilon$.*

Proof. We introduce the following spaces and operators, according to our previous notations.

- $H_1 = H_3 = L^2(\Omega)\times L^2(\Omega)$, $H_2 = H_4 = H_5 = L^2(\Omega)$.
- $A(y^0, z^T) = (y(T), z(T) - z^T)$, where (y,z) is the corresponding solution of (3.5) for $h = 0$. It will be convenient in what follows to decompose A as $A(y^0, z^T) = C(y^0, z^T) + D(y^0, z^T)$, where $C(y^0, z^T) = (y(T), z(T))$ and $D(y^0, z^T) = -z^T$. We also introduce $C_1(y^0, z^T) = y(T)$ and $C_2(y^0, z^T) = z(T)$.
- $B(h) = -(y(T), z(T))$, where (y,z) is the corresponding solution of (3.5) for $y^0 = 0$ (so, by the Duhamel formula, B is simply $-\Phi_T$, with Φ_T given in (3.4)).

By the superposition principle for linear equations, we have $y(T) = A_1(y^0, z^T) - B_1(h)$ and $z(T) = A_2(y^0, z^T) - B_2(h)$. So we can reformulate our question as follows: for any $(y^0, z^T) \in L^2(\Omega)^2$ and any $\varepsilon > 0$, there exists $h \in H_2$ such that the solution (y,z) of (3.5) satisfies $A_1(y^0, z^T) = B_1(h)$ and $||A_2(y^0, z^T) - B_2(h)||_{L^2(\Omega)} \leqslant \varepsilon$.

We first compute the adjoint operators of A_1, B_1 and of A, B (from which we can easily deduce the expression of A_2^* and B_2^*, but they are not explicitly needed). It will be convenient to introduce the adjoint system of (3.5):

$$
\begin{cases}
-\dfrac{\partial \phi}{\partial t} - \Delta \phi & = & \psi & \text{in } (0,T)\times\Omega, \\
\phi & = & 0 & \text{on } (0,T)\times\partial\Omega, \\
\phi\,(T,\cdot) & = & \phi^T & \text{in } \Omega, \\
-\dfrac{\partial \psi}{\partial t} - \Delta \psi & = & 0 & \text{in } (0,T)\times\Omega, \\
\psi & = & 0 & \text{on } (0,T)\times\partial\Omega, \\
\psi\,(T,\cdot) & = & \psi^T & \text{in } \Omega,
\end{cases}
\tag{3.6}
$$

where $\phi^T, \psi^T \in L^2(\Omega)$.

We will also need the following scalar equation:

$$
\begin{cases}
-\dfrac{\partial \eta}{\partial t} - \Delta \eta & = & 0 & \text{in } (0,T)\times\Omega, \\
\eta & = & 0 & \text{on } (0,T)\times\partial\Omega, \\
\eta\,(T,\cdot) & = & \phi^T & \text{in } \Omega,
\end{cases}
\tag{3.7}
$$

where $\phi^T \in L^2(\Omega)$. Then, reasoning by density and performing integrations by parts, the solution of (3.5) satisfies: for any $\phi^T, \psi^T \in L^2(\Omega)$,

$$
\langle y(T), \phi^T\rangle_{L^2(\Omega)} + \langle z(T), \psi^T\rangle_{L^2(\Omega)} - \langle y^0, \phi(0)\rangle_{L^2(\Omega)} = \int_0^T \int_\omega h(t,x)\phi(t,x)dxdt,
$$

where (ϕ, ψ) satisfies (3.6). In particular, taking $\psi^T = 0$ and $h = 0$ leads to

$$
\langle y(T), \phi^T\rangle_{L^2(\Omega)} = \langle y^0, \phi(0)\rangle_{L^2(\Omega)}.
$$

Since $\psi^T = 0$, we have $\psi = 0$, which implies that in fact this ϕ is equal to the solution η of (3.7). Hence,

$$
C_1^*(\phi^T) = \eta(0).
$$

Since $D^* = D$, we deduce that we also have

$$
A_1^*(\phi^T) = \eta(0).
$$

Now, taking $h = 0$ leads to

$$
\langle z(T), \psi^T\rangle_{L^2(\Omega)} + \langle y(T), \phi^T\rangle_{L^2(\Omega)} = \langle y^0, \phi(0)\rangle_{L^2(\Omega)}.
$$

Since $D^* = D$, we deduce that

$$
A^*(\phi^T, \psi^T) = (\phi(0), \psi^T).
$$

Applying a similar argument yields

$$
B_1^*(\phi^T, \psi^T) = -\eta \mathbb{1}_\omega
$$

and

$$
B^*(\phi^T, \psi^T) = -\phi \mathbb{1}_\omega.
$$

In particular, we observe that

$$
B_1^*(\phi^T, \psi^T) = B^*(\phi^T, 0). \tag{3.8}
$$

Now, our goal is to apply Proposition 2.4. $\mathcal{N}(B^*) = 0$ is true: if $B^*(\phi^T, \psi^T) = 0$, then $\phi = 0$ on $(0,T)\times\omega$. Differentiating and using (3.6), we obtain that $-\dfrac{\partial \phi}{\partial t} - \Delta\phi = \psi = 0$ on $(0,T)\times\omega$.

By Holmgren's uniqueness theorem, ψ is analytic in space, so $\psi = 0$ on $(0,T) \times \Omega$. We notably deduce by continuity in time that $\psi^T = 0$. Going back to the first equation in (3.6) and repeating the same reasoning implies that $\phi^T = 0$, as needed.

Introduce the following Hilbertian norm on $L^2(\Omega)^2$:

$$||(\phi^T, \psi^T)||_{**}^2 = \int_0^T \int_\omega \eta(t,x)^2 dxdt + \int_\Omega \psi^T(x)^2 dx,$$

and consider B^* to be the extension of B^* on the completed space X. Our goal is to prove that $\mathcal{N}(B^*) = 0$.

To prove this, we use the Carleman estimate proved for (3.7) in [1, Theorem 3], which implies that for any $(\phi^T, \psi^T) \in L^2(\Omega)^2$, we have, for some $c > 0$ (that does not depend on T and may change from line to line in what follows) and some $C(T) > 0$ (that might depend on T and might change from line to line),

$$\begin{aligned}\int_0^T \int_\Omega e^{-\frac{c}{T-t}} (\phi(t,x)^2 + \psi(t,x)^2) dxdt &\leqslant C(T) \int_0^T \int_\omega \phi(t,x)^2 dxdt \\ &= C(T) ||B^*(\phi^T, \psi^T)||_{L^2(\Omega)}^2 \\ &= ||(\phi^T, \psi^T)||_{R'}^2,\end{aligned}$$

which can be rewritten as

$$\int_0^T \int_\Omega e^{-\frac{c}{T-t}} ||\mathbb{T}_t^*(\phi^T, \psi^T)||_{L^2(\Omega)^2}^2 dt \leqslant C(T) ||(\phi^T, \psi^T)||_{R'}^2.$$

Since $(\tilde{\mathbb{T}}^*{}_t)_{t \geqslant 0}$ generates a strongly continuous semigroup on R' that coincides with $(\mathbb{T}^*{}_t)_{t \geqslant 0}$ on $\mathcal{H}$, we deduce by density of $\mathcal{H}$ in R' that for any $(\phi^T, \psi^T) \in R'$, we have

$$\int_0^T \int_\Omega e^{-\frac{c}{T-t}} ||\tilde{\mathbb{T}}^*{}_t(\phi^T, \psi^T)||_{L^2(\Omega)^2}^2 dt \leqslant C(T) ||(\phi^T, \psi^T)||_{R'}^2. \tag{3.9}$$

Moreover, using (3.8) and the fact that B^* is linear continuous, for $(\phi^T, \psi^T) \in \mathcal{H}$, we have

$$\begin{aligned}||B^*(\phi^T, \psi^T)||_{L^2(\Omega)}^2 &= ||B^*(\phi^T, 0) + B^*(0, \psi^T)||_{L^2(\Omega)}^2 \\ &\leqslant c \left(||B_1^*(\phi^T, \psi^T)||_{L^2(\Omega)}^2 + ||\psi^T||_{L^2(\Omega)}^2 \right) \\ &\leqslant c ||(\phi^T, \psi^T)||_{**}^2.\end{aligned}$$

Hence, by density, if $(\phi^T, \psi^T) \in X$, then, $(\phi^T, \psi^T) \in R'$ and

$$||(\phi^T, \psi^T)||_{R'}^2 \leqslant c ||(\phi^T, \psi^T)||_{**}^2. \tag{3.10}$$

We can now conclude. If $B^*(\phi^T, \psi^T) = 0$ with $(\phi^T, \psi^T) \in X$, then, $(\phi^T, \psi^T) \in R'$ by (3.10), and by (3.9), for any $t \in (0,T)$, we have that $\tilde{\mathbb{T}}_t^*(\phi^T, \psi^T) = 0$ in $\mathcal{H}$, so $\tilde{\mathbb{T}}_t^*(\phi^T, \psi^T) = 0$ also in R'. Since $(\tilde{\mathbb{T}}_t^*)_{t \geqslant 0}$ generates a strongly continuous semigroup on R', we also deduce that $(\phi^T, \psi^T) = 0$, as wanted. □

Remark 3.2. *In fact, Theorem 3.1 also implies the same property for system* (3.3)*, namely, For any* $(y^0, z^0, z^T) \in L^2(\Omega)^3$ *and any* $\varepsilon > 0$*, there exists* $h \in H_2$ *such that the solution* (y, z) *of* (3.3) *satisfies* $y(T) = 0$ *and* $||z(T) - z^T||_{L^2(\Omega)} \leqslant \varepsilon$*. Indeed, introducing* $A_3(z^0)$ *the solution of* (3.3) *with* $y^0 = h = 0$*, and with the notations of the proof of Theorem 3.1, the superposition principle says that our problem is equivalent to: for any* $(y^0, z^0, z^T) \in L^2(\Omega)^3$ *and any* $\varepsilon > 0$*, there exists* $h \in H_2$ *such that* $A_1(y^0, z^T) = B_1(h)$ *and* $||A_2(y^0, z^T) + A_3(z^0) - B_2(h) - z^T|_{L^2(\Omega)}| \leqslant \varepsilon$*, which clearly follows from Theorem 3.1 (just change* z^T *into* $z_T - A_3(z^0)$*).*

4 Further results, comments, open problems

More products. Of course, one can consider operators with codomain in products of more than two spaces: assume that we have some Hilbert spaces $E_1, \dots E_p$ on which one wants the "exact range inclusion property", and $F_1, \dots F_m$ on which one wants the "range-closure inclusion" property. One can obtain the analogue of Theorem 2.1 just by setting $H_4 = E_1 \times \dots \times E_p$ and $H_5 = F_1 \times \dots \times F_m$, endowed with the natural Hilbertian product norm.

The case of Banach spaces. The analog of Douglas' majorization and factorization theorem in Banach spaces might be false if we work with non-reflexive Banach spaces (see [6]). However, the result given in [6] notably gives that we have an analog of Douglas' majorization and factorization theorem for reflexive Banach spaces, for which all orthogonality relations that are true in Hilbert spaces also hold. So Theorem 2.1 is also valid in the case of reflexive Banach spaces.

Extending Proposition 2.4. Even if the condition $\mathcal{N}(B^*) = 0$ is often verified in practice in many examples (notably coming from control theory), it is a very restrictive condition. Here is a possible way to extend a little bit the results. Assume that $\overline{\mathcal{R}(B)}$ is itself a cartesian subspace of $H_4 \times H_5$. Then, we can replace B with $\tilde{B}$, obtained by restricting the codomain to $\overline{\mathcal{R}(B)}$. $\tilde{B}$ has now dense range, so $\mathcal{N}(\tilde{B}^*) = 0$. Since $\overline{\mathcal{R}(B)}$ is cartesian, we can apply Proposition 2.4 to this operator. In the general case, we have no idea of how to find an analogue to Proposition 2.4. In particular, it would be reasonable to expect a condition on A_2 to appear.

Unbounded operators. We do not treat the general case of unbounded operators A and B on Hilbert or Banach spaces, that is more involved (see [5] or [10]). Here, the situation is expected to be even more involved and we leave it for further investigation.

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